Math 265 Midterm 2 Review

March 16, 2016

1 Things you should be able to do

This list is not meant to be exhaustive, but to remind you of things I may ask you to do on the exam. These are roughly in the order they appear in the book. You should certainly go through your homework problems, problems we did in class (especially from the handouts), the review problems in the next section, and the multiple choice practice that I posted on the course page.

- Determine whether a set is a subspace of one of the standard vector spaces we have discussed ($\mathbb{R}^n$, P_n, M_{mn}).
- Understand the null space of a matrix, why it is a subspace of $\mathbb{R}^n$ and determine whether a vector belongs to the null space of a matrix.
- Understand the properties that make a space a vector space. (Properties a, b, and 1-8.)
- Span: you should be able to define the span of a set of vectors, be able to tell when a set of vectors is a spanning set for a given subspace and be able to tell when a vector is the span of some given set of vectors.
- Linear independence and dependence: You should be able to define what it means for a set of vectors to be linearly independent or dependent. You should be able to use various methods to determine whether a set of vectors is linearly dependent or linearly independent. You should understand that if the zero vector is in a set of vectors, then that set is linearly dependent and why that is the case.

- Basis and Dimension: You should be able to define basis and dimension of a vector space. You should be able to produce a basis when you are given a spanning set for a vector space or subspace and use that basis to conclude what the dimension of the space is. You should be able to produce a basis for a subspace even when you are not given a spanning set to start out with.

- Col A, Row A and Null A along with rank and nullity: You should be able to define what the column space, row space and null space of a matrix are. You should be able to define the rank and nullity of a matrix and understand why rank $A+$ nullity $A = n$ (where A is an m x n matrix). You should understand the identities dealing with rank and nullity and the transpose of a matrix. You should be able to tell me the possible range of values for the rank of a matrix when I give you its size. In general, you should be able to say something about the RREF of the matrix when I give you the rank. You should be able to find a basis for Col A, Row A and Null A.

- You should understand the implications when rank $A =$ n, especially when A is an n x n matrix (A is then invertible, row equivalent to the identity, etc.). You should also understand the what this tells us about the column and row vectors of A (linearly independent).

2 Questions to test your conceptual understanding

Classify each of the following statements as TRUE or FALSE:

1. If $A\mathbf{x} = \mathbf{0}$ has a unique solution, then there are no vectors in Null A.
2. A square matrix A is nonsingular if and only if det $A = 0$.
3. Let V be a subspace of $\mathbb{R}^n$ of dimension d and let S be a set of m vectors in V. If $m < d$ then S cannot span V.
4. Let V be a subspace of $\mathbb{R}^n$ of dimension d and let S be a set of m vectors in V. If $m > d$ then S must span V.
5. Let V be a subspace of $\mathbb{R}^n$ of dimension d and let S be a set of m vectors in V. If $m < d$ then S is linearly independent.
6. Let V be a subspace of $\mathbb{R}^n$ of dimension d and let S be a set of m vectors in V. If $m = d$ and S spans V, then S is a basis for V.

7. If A is an m x n matrix and $n > m$, then the null space of A is not $\{\mathbf{0}\}$.

8. If A is an m x n matrix, then dim Null A + dim Col $A = n$.

9. If A is an m x n and rank $A < n$, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

10. The nullity of A always equals the nullity of A^T.

11. The rank of A always equals the rank of A^T.

12. If A is an m x n matrix, then dim Null A^T + dim Col $A = m$.

Which of the following are subspaces of $\mathbb{R}^3$?

1. The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $y = x^2$.

2. The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x \geq 0$.

3. The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $z = 0$.

4. The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $3x - 2y - z = 0$.

5. The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x + y = 3$.

Which of the following are subspaces of M_{22}?

1. The set of 2 x 2 symmetric matrices.

2. The set of 2 x 2 matrices with zero trace.

3. The set of 2 x 2 matrices with zero determinant.

4. The set of 2 x 2 nonsingular matrices.

5. The set of 2 x 2 diagonal matrices.

6. The set of 2 x 2 lower triangular matrices.

1. Suppose that $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Is the span of the set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ all of $\mathbb{R}^3$? Justify your answer.

2. Is $A = \begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \right\}$? Are these four matrices linearly dependent or linearly independent?

3. Are the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ linearly dependent or independent?

4. For what values of c are the vectors $\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix}$ linearly independent?

5. Find a basis for span$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ where

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -5 \end{bmatrix}, \mathbf{w}_4 = \begin{bmatrix} 2 \\ -2 \\ 3 \\ 0 \end{bmatrix}.$$

6. Find a basis for all 2 x 2 symmetric matrices.

7. Find a basis for the subspace $W = \text{span}\{t^3+t^2+2t+1, t^3-3t+1, t^2+t+2, t+1, t^3+1\}$ of P_3.

8. Find a basis for the subspace $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 3x+5y+7z=0 \right\}$ of $\mathbb{R}^3$. What is dim W?

9. Find a basis for $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \right\}$.

Hint: The RREF of $\begin{bmatrix} 1 & 3 & 11 & 7 \\ 2 & 2 & 10 & 6 \\ 2 & 1 & 7 & 4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

10. Find the rank and nulity of the following matrix:

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 6 & -8 & 1 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

11. Is A is a 5 x 7 matrix, what are the possible values for rank A? If A is a 7 x 3 matrix, what are the possible values of rank A? If A is a 4 x 6 matrix, what are the possible values for nullity A?

12. Given that A has RREF R, where

$$A = \begin{bmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

find bases for the row space of A, the column space of A and the null space of A. Give the rank and nullity of A and A^T.

13. If A is a 5 x 7 matrix and nulllity $A = 3$, what is nullity A^T?

14. If A is an 4 x 4 matrix where rank $A = 4$, what can you say about det A? Do the columns of A form a basis for $\mathbb{R}^4$? How many non-zero rows are there in the RREF of A? Does the system $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b} \in \mathbb{R}^4$? If so, is the solution unique?

15. If A is a 6 x 6 matrix where rank $A = 4$, what can you say about det A? Do the rows of A form a basis for $\mathbb{R}_4$? What is the dimension of Col A^T? How many vectors are there in any basis for Null A? How many solutions are there to the homogeneous system $A\mathbf{x} = \mathbf{0}$?

16. If A is a 4 x 4 matrix and det $A = -3$ what is rank A?

17. If A is a 3 x 3 matrix and det $A = 0$, can you determine whether the column vectors of A form a basis for $\mathbb{R}^3$? What are the possible values for rank A?