Math 265
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1/28/16

Class Handout #5

Lemma 2.1: If A is an $n \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\mathbf{x} = \mathbf{0}$, then A is row equivalent to I_n.

Theorem 2.8: A is invertible if and only if A is the product of elementary matrices.

Corolloary 2.2: A is invertible if and only if A is row equivalent to I_n.

So we have shown:

Theorem 2.9: If A is $n \times n$, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if A is a singular (noninvertible) matrix (i.e. the RREF of A is not I_n).

Theorem 2.10: An $n \times n$ matrix A is singular is and only if the reduced row echelon form of A has a row of zeros.

What we have shown is that the following statements are *equivalent* for an nxn matrix A:

1. A is invertible (nonsingular).
2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. A is row equivalent to I_n. (The RREF of A is I_n.)
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every n-vector $\mathbf{b}$.
5. A is a product of elementary matrices.

Finding A^{-1}: Form the partitioned (double) matrix $[A \mid I_n]$ and perform elementary row operations to reduce the matrix to $[I_n \mid A^{-1}]$, if possible.

Note: In order to find A^{-1}, we don't have to determine in advance whether or not it exists. We simply start to reduce the partitioned matrix $[A \mid I_n]$ to RREF obtaining $[C \mid D]$. If $C = I_n$ then A is invertible and $A^{-1} = D$. Otherwise, $C \neq I_n$, so C has a row of zeros and A is noninvertible.

We can also now prove:

Theorem 2.11: If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$. Thus, $B = A^{-1}$.

Section 3.1: Determinants

Definition: Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant function, denoted by det, is defined by

$$\det(A) = \sum \pm a_{1j_1} a_{2j_2} \cdots a_{nj_n},$$

where the summation is over all permutations $j_1 j_2 \cdots j_n$ of the set $\{1, 2, \ldots, n\}$. The sign is $+$ for an even permutation and $-$ for an odd permutation.

Section 3.3: Cofactor Expansion:

Definition: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and the jth column. Then $\det M_{ij}$ is called the minor of a_{ij}.

Definition: The cofactor A_{ij} of a_{ij} is defined as $A_{ij} = (-1)^{i+j} \det M_{ij}$.

Theorem 3.10: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then,

$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$ *(cofactor expansion along the ith row)*, and also

$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$ *(cofactor expansion along the jth column).*