Math 265
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Class Handout #4

Note: If the row echelon form of an augmented matrix representing a system does not have a leading one in every column, always let the variables corresponding to columns without leading ones be the variables you solve in terms of (the ones you set to be free parameters).

Theorem 2.4: *(The more unknowns theorem)* A homogeneous system of m linear equations in n unknowns always has a nontrivial solution if $m < n$, that is, if the number of variables exceeds the number of equations.

Section 2.3: Elementary Matrices; Finding A^{-1}

Definition: An $n \times n$ **elementary matrix** is a matrix obtained from I_n by performing a single elementary row operation.

Example 1:

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2: Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 3 & 0 & 2 \end{bmatrix}$ and let $B = A_{-3r_1+r_3 \to r_3} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 0 & -6 & -1 \end{bmatrix}$. Now let $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$, the 3×3 elementary matrix representing $-3r_1 + r_3 \to r_3$. What is EA?

Theorem 2.6: If A and B are $m \times n$ matrices, then A is row equivalent to B if and only if there exist $\underline{m \times m}$ elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = B$.

Theorem 2.7: An elementary matrix is always invertible and its inverse is an elementary matrix of the same type.

Lemma 2.1: If A is an $n \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\mathbf{x} = \mathbf{0}$, then A is row equivalent to I_n.

Theorem 2.8: A is invertible if and only if A is the product of elementary matrices.

Corolloary 2.2: A is invertible if and only if A is row equivalent to I_n.

So we have shown:

Theorem 2.9: If A is $n \times n$, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if A is a singular (noninvertible) matrix (i.e. the RREF of A is not I_n).

What we have shown is that the following statements are *equivalent* for an nxn matrix A:

1. A is invertible (nonsingular).
2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. A is row equivalent to I_n. (The RREF of A is I_n.)
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every n-vector $\mathbf{b}$.
5. A is a product of elementary matrices.

Matrix Inverses

In order to find A^{-1}, we don't have to determine in advance whether or not it exists. We simply start to reduce the partitioned matrix $[A \; I_n]$ to RREF obtaining $[C \; D]$. If $C = I_n$ then A is invertible and $A^{-1} = D$. Otherwise, $C \neq I_n$, so C has a row of zeros and A is noninvertible.