Math 265
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Class Handout #20

Theorem: If A is symmetric and λ_0 is an eigenvalue of A of multiplicity k, then nullity $(A - \lambda_0 I_n) = k$. Therefore, a symmetric matrix A is always diagonalizable.

Theorem 7.7: If A is a symmetric matrix, then eigenvectors associated to distinct eigenvalues of A are *orthogonal* (not just linearly indepedent).

Definition: A real square matrix A is called *orthogonal* if $A^T A = I_n$, that is $A^{-1} = A^T$.

Theorem 7.8: An n x n matrix A is orthogonal if and only if the columns of A form an orthonormal set.

Exercise 1: If P is an orthogonal matrix, do the columns of P form an orthonormal basis for $\mathbb{R}^n$? Do the rows of P form an orthonormal basis for $\mathbb{R}_n$?

Exercise 2: Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$. Find the eigenvalues of A. Then find a basis for each eigenspace.

Theorem 7.9: If A is a symmetric n x n matrix, then there exists an orthogonal matrix P such that $A = PDP^{-1} = PDP^T$, where the eigenvalues of A lie along the main diagonal of D.

Procedure for diagonalizing matrices:

1. Find the eigenvalues of A, that is find the roots of the characteristic polynomial of A. If there are no complex eigenvalues then proceed to step 2. If there are any complex eigenvalues then A is not diagonalizable.

2. Find the dimensions of the corresponding eigenspaces by finding dim Null$(A - \lambda_0 I_n)$ = nullity $(A - \lambda_0 I_n)$. If nullity $(A - \lambda_0 I_n)$ is equal to the multiplicity of λ_0 for each eigenvalue λ_0 then A is diagonalizable.

3. To diagonalize A, start by finding a basis of eigenvectors for the eigenspace corresponding to each eigenvalue. That is, find a basis for Null$(A - \lambda_0 I_n)$.

4. If A is NOT symmetric, form the invertible matrix P whose column vectors are the eigenvectors you found in step 3. Form the diagonal matrix D whose entries along the main diagonal are the corresponding eigenvalues (the order of the values is determined by how you put down the eigenvectors as the columns of P).

5. If A IS symmetric, the eigenvectors coming from different eigenvalues are orthogonal, but you must convert the basis for each eigenspace from step 3 into an orthonormal basis. For this, you apply the Gram-Schmidt process to produce an ORTHONORMAL basis for each eigenspace. Note: When you have a single eigenvector in your basis for your eigenspace, you must still normalize it to be a unit vector!

6. Now form the orthogonal matrix P whose columns are the unit vectors in the orthonormal bases you produced in step 4. Then form the diagonal matrix D whose entries on the main diagonal are the eigenvalues of A, where the order is again determined by the order in which you placed the eiegenvectors into the matrix P.

Systems of differentiable equations:

Exercise 3: Find the general solution for the diagonal system:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Exercise 4: Find the particular solution to the diagonal system above with the initial condition $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$.

Theorem 8.8: If the $n \times n$ matrix A is diagonalizable, with eigenvectors $\mathbf{p}_1, \ldots, \mathbf{p}_n$ associated to eigenvalues $\lambda_1, \ldots, \lambda_n$, then the general solution to the linear system of differential equations $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = b_1\, \mathbf{p}_1\, e^{\lambda_1 t} + b_2\, \mathbf{p}_2\, e^{\lambda_2 t} + \cdots + b_n\, \mathbf{p}_n\, e^{\lambda_n t}.$$

Exercise 5: Find the general solution for the system:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -14 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Exercise 6: Find the particular solution to the diagonal system above with the initial condition $\mathbf{x}(0) = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$.