Math 265
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Class Handout #19

Complex Numbers:

$\mathbb{C}$ is the set of all numbers of the form $z = a + bi$ where $i^2 = -1$ and $a, b \in \mathbb{R}$. a is the real part of z and b is the imaginary part of z.

Complex conjugation: $\overline{z} = \overline{a + bi} = a - bi$

Properties of complex conjugation:

1. $\overline{\overline{z}} = z$
2. $\overline{z + w} = \overline{z} + \overline{w}$
3. $\overline{zw} = \overline{z}\overline{w}$
4. c is a real number if and only if $c = \overline{c}$
5. $z\overline{z}$ is always a nonnegative real number and $z\overline{z} = 0$ if and only if $z = 0$

Properties of complex matrix conjugation:

1. $\overline{\overline{A}} = A$
2. $\overline{A + B} = \overline{A} + \overline{B}$
3. $\overline{AB} = \overline{A}\overline{B}$
4. c is any real or complex number $\overline{cA} = \overline{c}\overline{A}$
5. $(\overline{A})^T = \overline{A^T}$
6. $(\overline{A})^{-1} = \overline{A^{-1}}$

Note: Determinants for square matrices with complex entries are computed in the same exact way that they are computed for matrices with real entries.

Shortcut for inverse of any real or complex 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $A^{-1} = \dfrac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Exercise 1: Let $A = \begin{bmatrix} 3i & -i \\ 1 & i \end{bmatrix}$ what is the a_{22} entry of A^{-1}?

Solving systems with complex entries:

Exercise 2: Solve the following system:

$$\begin{cases} (1+i)x_1 + & (2+i)x_2 & = & 5 \\ (2-2i)x_1 + & ix_2 & = & 1+2i \end{cases}$$

Exercise 3: (Alternate method) Solve the following system:

$$\begin{cases} (2+i)x_1 + & (1+i)x_2 & = & 5 \\ (3-i)x_1 + & (2-2i)x_2 & = & 1+2i \end{cases}$$

Complex Vector Spaces:

Exercise 4: (Linear dependence, span, etc.)

Let V be the complex vector space $\mathbb{C}^3$. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} i \\ 0 \\ 1_i \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

- Determine whether $\mathbf{v} = \begin{bmatrix} -1 \\ -3+3i \\ -4+i \end{bmatrix}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.
- Determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{C}^3$.
- Determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent subset of $\mathbb{C}^3$.

Recall that a matrix A is symmetric if $A = A^T$. Note: This means A is necessarily a square matrix.

Theorem 7.6: All of the root of the characteristic polynomial $p_A(t)$ of a symmetric matrix A are real numbers.

Exercise 5: Consider the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 3 \end{bmatrix}$ with eigenvalues $\lambda_1 = -2$, $\lambda_2 = 4$ and $\lambda_3 = -1$. Notice that A is symmetric. Find a basis for the eigenspace corresponding to each eigenvalue.

Theorem: If A is symmetric and λ_0 is an eigenvalue of A of multiplicity k, then nullity $(A - \lambda_0 I_n) = k$. Therefore, a symmetric matrix A is always diagonalizable.

Theorem 7.7: If A is a symmetric matrix, then eigenvectors associated to distinct eigenvalues of A are *orthogonal* (not just linearly indepedent).

Definition: A real square matrix A is called *orthogonal* if $A^T A = I_n$, that is $A^{-1} = A^T$.

Exercise 6: Verify that the following matrix is orthogonal: $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Theorem 7.8: An n x n matrix A is orthogonal if and only if the columns of A form an orthonormal set.

Theorem 7.9: If A is a symmetric n x n matrix, then there exists an orthogonal matrix P such that $A = PDP^{-1} = PDP^T$, where the eigenvalues of A lie along the main diagonal of D.

Exercise 7: Let A be the matrix from exercise 2. Find an orthogonal matrix P such that $A = PDP^T$.