Math 265
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4/7/16

Class Handout #17

The nice thing about linear transformations is that once you know $L(\mathbf{u})$ and $L(\mathbf{v})$ you know how L transforms any linear combination of $\mathbf{u}$ and $\mathbf{v}$. For example, if $L(\mathbf{u}) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $L(\mathbf{v}) = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}$, what is $L(3\mathbf{u} - 2\mathbf{v})$?

What this means is that is $L : V \longrightarrow W$ is a linear transformation, and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is a basis for V, then once we know $L(\mathbf{v}_1), \dots, L(\mathbf{v}_n)$, we know $L(\mathbf{v})$ for any $\mathbf{v} \in V$.

Let's examine a special case of this. Let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Then for a random vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$, we know that $L\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = v_1 L\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + v_2 L\left(\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}\right) + \cdots + v_n L\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right)$.

This means $L(\mathbf{v}) = A\mathbf{v}$ where $A =$

The matrix A above is called the *standard matrix* for L.

Theorem 6.3: Let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and let A be the m x n matrix whose jth column is $L(\mathbf{e_j})$. Then A, which we call the standard matrix for L, is the unique matrix with the property that $A\mathbf{x} = L(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$.

Exercise 1: Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 - 2x_3 \end{bmatrix}$. Find the standard matrix A for L.

Exercise 2: Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by $L(\mathbf{u}) = 5\mathbf{u}$. Find the standard matrix A for L. This sort of linear transformation is called a *dilation*.

Eigenvalues and eigenvectors:

Definition: Let A be an n x n matrix (the fact that A is square is important). A **nonzero** vector $\mathbf{x} \in \mathbb{R}^n$ is called an *eigenvector* of A is $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is called an *eigenvalue* of A.

Example: Let $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{bmatrix}$. Verify that $\mathbf{x}$ is an eigenvector of A and find the corresponding eigenvalue.

Theorem 7.1: Let A be an n x n matrix. The eigenvalues of A are the roots of the characteristic polynomial of A (that is they are precisely the solutions of the characteristic equation $\det(A - \lambda I_n) = 0$).

Procedure for finding eigenvalues and corresponding eigenvectors:
Step 1: Determine the roots of the characteristic $p(\lambda) = \det(A - \lambda I_n)$. These are the eigenvalues of A.

Step 2: For each root λ_0, find all nontrivial solutions to the homogeneous system

$$(A - \lambda_0 I_n)\mathbf{x} = \mathbf{0}.$$

By solving for the null space of the matrix $(A - \lambda_0 I_n)$ you can find a basis for the λ_0–eigenspace.

Exercise 3: Find all eigenvalues and a basis for each corresponding eigenspace of

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & -3 \end{bmatrix}.$$

Exercise 4: Find the eigenvalues for $A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$.