Math 265
Professor Priyam Patel
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Class Handout #14

Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$

Exercise 1:

What are the lengths of $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \in \mathbb{R}^2$ and $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \in \mathbb{R}^3$?

What is the distance between the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}$?

Properties of the Dot Product/Standard Inner Product on $\mathbb{R}^n$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. The standard inner product on $\mathbb{R}^n$ has the following properties:

1. $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}; \mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ iff $\mathbf{u} = \mathbf{0}$
2. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
4. $c\mathbf{u} \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot c\mathbf{v}$

Exercise 2:

Calculate the angles between the following pairs of vectors: $\mathbf{u_1} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v_1} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, and $\mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular) iff $\mathbf{u} \cdot \mathbf{v} = 0$.

A *unit vector* is a vector of length 1.

Definition: A set of vectors S in $\mathbb{R}^n$ (or $\mathbb{R}_n$) is called an *orthogonal set* if any two distinct vectors in S are orthogonal, that is, the set of vectors is pairwise orthogonal. If, in addition, each vector in S is a unit vector, then S is called an *orthonormal set.*

Example: Standard basis in $\mathbb{R}^n$ is an orthonormal set with respect to the standard inner product (dot product).

Note: If S consists of k **non-zero** vectors and is orthogonal, we can always produce an orthonormal set of k vectors from S.

Exercise 3: Which of the following sets of vectors are orthogonal, orthonormal or neither? For those that are orthogonal or orthonomal, is the set linearly independent? If a set below is orthogonal, but not orthonormal, produce the related orthonormal set of vectors if you can.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$S_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$S_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise 4: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ -2 \\ 3 \end{bmatrix}$. For what value of a are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

Theorem 5.4: Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthogonal set of **non-zero** vectors in $\mathbb{R}^n$ or $\mathbb{R}_n$. Then S is linearly independent.

Section 5.3 Summarized:

Why do we call the dot product the "standard" inner product? Because any operation $(\mathbf{u}, \mathbf{v})$ on vectors in vector space V that satisfy the properties that the dot product satisfies is called an inner product.

Example: In $\mathbb{R}^2$ let $(\mathbf{u}, \mathbf{v}) = u_1v_1 - u_2v_1 - u_1v_2 + 3u_2v_2$. Then this operation on pairs of vectors is an inner product on $\mathbb{R}^2$.

Example: In the the vector space V of continuous real-valued functions, let

$$(f, g) = \int_0^1 f(t)g(t)dt.$$

This is an inner product on V.

Section 5.4:

Definitions: An orthogonal set of vectors S that is also a basis for a subspace of $\mathbb{R}^n$ is called an *orthogonal basis* for that subspace. Likewise, an orthonormal set of vectors S that is also a basis for a subspace of $\mathbb{R}^n$ is called an *orthonormal basis* for that subspace.

Theorem 5.5: Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a an orthogonal basis for a subspace W and let $\mathbf{u}$ be any vector in W. Then, $\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k}$ where

$$c_i = \frac{\mathbf{u} \cdot \mathbf{v_i}}{\mathbf{v_i} \cdot \mathbf{v_i}} = \frac{\mathbf{u} \cdot \mathbf{v_i}}{\|v_i\|^2}.$$

If S is an orthonormal basis, then $\|v_i\|^2 = 1$ for all i and

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{v_1})\mathbf{v_1} + (\mathbf{u} \cdot \mathbf{v_2})\mathbf{v_2} + \cdots + (\mathbf{u} \cdot \mathbf{v_k})\mathbf{v_k}.$$

Exercise 5: Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$. Verify that $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthogonal set. Therefore, S is a basis for $\mathbb{R}^3$.

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Find the coefficients c_1, c_2, c_3 in $\mathbf{u} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$.