MATH 351 FINAL REVIEW – PROBLEMS

1. Practice Questions from Textbook

Section 1.1 TF: 1 - 10 Exs: 3, 4, 5 , 10, 12, 13, 14, 16, 17, 20, 24, 26, 27, 31, 32, 34a, d,e, 35, 36
Section 1.2 TF: 1, 2, 3, 4, 5, 6 Exs: 1-8
Section 1.3 TF: 1, 2, 4 Exs: 1, 2, 3a, b, c, d, 5, 7, 8, 20, 21, 22
Section 1.4 TF: 1, 2, 3, 4, 9, 10, 12 Exs: 1, 4, 11, 12, 27, 28
Section 2.1 TF: 2, 3, 4 Exs: 1, 3, 5
Section 2.2 TF: 1, 2,4, 8, 9 Exs: 1, 5, 7, 8, 10, 11, 12, 14, 15, 16, 18, 19, 21
Section 2.3 TF: 4, 5, 6, 8 Exs: 2, 3, 5, 12, 21, 22
Section 3.1 TF: 1, 6 Exs: 14, 15, 20, 27
Section 3.2 TF: 5, 7 Exs: 1, 3, 26, 29
Section 3.3 TF: 1, 8 Exs: 3, 10, 18, 20, 21, 22, 22, 23
Section 3.5 TF: 1, 2, 8 Exs: 4a-d, 8
Section 4.1 TF: 1-5 Exs: 1, 9
Section 4.2 TF: 1, 4 Exs: 1, 5, 15, 16
Section 5.1 TF: 1, 2, 5, 6, 7, 8, 9, 11 Exs: 1, 6, 9, 15
Section 5.2 TF: 1 Exs: 2, 4, 5, 12
Section 5.3 Exs: 1, 3, 15
Section 6.1 TF: 1, 2, 3, 9 Exs: 1, 2, 3, 9
Section 6.2 TF: 1, 4 Exs: 1, 3a-d, 5, 12

2. Extra Questions

(1) Suppose that $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & -1 \\ -1 & 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 \\ 3 & -3 \\ 4 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 6 & 2 \\ -3 & 0 & 4 \end{bmatrix}$.

Which of the following quantities are defined? Calculate those that are.

a. $-3D$
b. $B + 2C$
c. $A + B$
d. $B^T + 2C$

(2) Is there a 2×2 matrix A such that the homogeneous system represented by $[A\ \mathbf{0}]$ has only $\mathbf{0}$ as a solution?

(3) Is there a 2×2 matrix A such that the homogeneous system represented by $[A\ \mathbf{0}]$ has infinitely many solutions?

(4) Is there a 2×2 matrix A such that the homogeneous system represented by $[A\ \mathbf{0}]$ has no solutions?

(5) For each of the following, give an example of a matrix satisfying the criteria, or explain why no such matrix exists:

- a 3 x 4 matrix R in RREF such that for *every* $\mathbf{c} \in \mathbb{R}^3$ the augmented matrix $[R\ \mathbf{c}]$ represents a consistent system.
- a 3 x 4 matrix R in RREF such that the the augmented matrix $[R\ \mathbf{0}]$ represents a system that has a *unique* solution.
- a 4 x 3 matrix R in RREF such that for *every* $\mathbf{c} \in \mathbb{R}^4$ the augmented matrix $[R\ \mathbf{c}]$ represents a system that has at least one solution.
- a 4 x 3 matrix R in RREF such that the the augmented matrix $[R\ \mathbf{0}]$ represents a system that has a *unique* solution $\mathbf{x} \in \mathbb{R}^3$
- a 4 x 4 matrix R in RREF such that such that the the augmented matrix $[R\ \mathbf{0}]$ represents a system that has a *nontrivial* solution $\mathbf{x} \in \mathbb{R}^4$

(6) Suppose that $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Is the span of the set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ all of $\mathbb{R}^3$? Justify your answer.

(7) Is $A = \begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \right\}$? Are these four matrices linearly dependent or linearly independent?

(8) Are the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ linearly dependent or independent?

(9) For what values of c are the vectors $\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix}$ linearly independent?

(10) Find a basis for span$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ where

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -5 \end{bmatrix}, \mathbf{w}_4 = \begin{bmatrix} 2 \\ -2 \\ 3 \\ 0 \end{bmatrix}.$$

(11) Find a basis for all 3 x 3 symmetric matrices.

(12) Find a basis for the subspace $W = \text{span}\{t^3+t^2+2t+1, t^3-3t+1, t^2+t+2, t+1, t^3+1\}$ of P_3.

(13) Find a basis for the subspace $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 3x + 5y + 7z = 0 \right\}$ of $\mathbb{R}^3$. What is dim W?

(14) Find a basis for $V = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \right\}$. What is the dimension of V?

(15) Find the rank and nulity of the following matrix:

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 6 & -8 & 1 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

(16) Is A is a 5 x 7 matrix, what are the possible values for rank A? If A is a 7 x 3 matrix, what are the possible values of rank A? If A is a 4 x 6 matrix, what are the possible values for nullity A?

(17) Given that A has RREF R, where

$$A = \begin{bmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

find bases for the row space of A, the column space of A and the null space of A. Give the rank and nullity of A and A^T.

(18) If A is a 5 x 7 matrix and nulllity $A = 3$, what is nullity A^T?

(19) Suppose A is an 4 x 4 matrix where rank $A = 4$. Do the columns of A form a basis for $\mathbb{R}^4$? How many non-zero rows are there in the RREF of A? Does the system $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b} \in \mathbb{R}^4$? If so, is the solution unique?

(20) Suppose A is a 6 x 6 matrix where rank $A = 4$. Do the rows of A form a basis for $\mathbb{R}_4$? What is the dimension of Col A^T? How many vectors are there in any basis for Null A? How many solutions are there to the homogeneous system $A\mathbf{x} = \mathbf{0}$?

(21) For matrices A, B, if AB is defined must BA also be defined?

(22) Is the matrix product AA^T always defined?

(23) Is every invertible matrix square? Is every square matrix invertible?

(24) Is the product of invertible matrices always invertible?

(25) What can you say about the reduced row echelon form of an invertible matrix?

(26) Suppose that

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & -1 \\ -1 & 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 \\ 3 & -3 \\ 4 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 6 & 2 \\ -3 & 0 & 4 \end{bmatrix}.$$

Which of the following are defined? Calculate those that are:

(a) AB (b) AD^T (c) BAC (d) CAB (e) C^TC

(27) What is the relationship between the determinant of a matrix and the determinant of its inverse?

(28) What is the relationship between the determinant of a matrix and the determinant of its transpose?

(29) Is det $(AB) = (\det A)(\det B)$? Is det $(A + B) = (\det A) + (\det B)$?

(30) If A is a 2×2 matrix, what is det $(-A)$? What if A is 3×3? If A is a 2×2 matrix, what is det $2A$? What if A is 3×3?

(31) If A is an 4 x 4 matrix where rank $A = 4$, what can you say about det A? Do the columns of A form a basis for $\mathbb{R}^4$? How many non-zero rows are there in the RREF of A? Does the system $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b} \in \mathbb{R}^4$? If so, is the solution unique?

(32) If A is a 4 x 4 matrix and det $A = -3$ what is rank A?

(33) If A is a 3 x 3 matrix and det $A = 0$, can you determine whether the column vectors of A form a basis for $\mathbb{R}^3$? What are the possible values for rank A?

(34) For what values of a and b is $\mathbf{v} = \begin{bmatrix} a \\ b \\ 2 \end{bmatrix}$ orthogonal to both $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

For those values of a and b does $\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$ form an orthogonal set?

(35) Find an orthogonal basis for the subspace of $\mathbb{R}^3$ consisting of all vectors of the form $\begin{bmatrix} a \\ a+b \\ b \end{bmatrix}$.

(36) Let $\mathcal{V}$ be the subspace of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$. For $Y = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$, find $\text{Proj}_{\mathcal{V}} Y$ and $\text{Orth}_{\mathcal{V}} Y$.

(37) Let $\mathcal{V} = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} \right\}$. Using the Gram-Schmidt process, find an orthogonal basis for $\mathcal{V}$.

Classify each of the following statements as TRUE or FALSE:

(1) $\det (A+B) = \det A + \det B$.
(2) $\det (AB) = \det (BA)$.
(3) $\det A^T = \det A$.
(4) $\det A^{-1} = \det A$.
(5) $\det (A+B) = \det (B+A)$.
(6) $(AB)^T = B^T A^T$.
(7) $(A+B)^T = A^T + B^T$.
(8) $(A^{-1})^T = (A^T)^{-1}$.
(9) AA^T is always symmetric.
(10) AA^T is always nonsingular.
(11) AA^T always equals $A^T A$.
(12) If $A\mathbf{x} = \mathbf{0}$ has a unique solution, then there are no vectors in Null A.
(13) A square matrix A is nonsingular if and only if $\det A = 0$.
(14) Let V be a subspace of $\mathbb{R}^n$ of dimension d and let S be a set of m vectors in V. If $m < d$ then S cannot span V.
(15) Let V be a subspace of $\mathbb{R}^n$ of dimension d and let S be a set of m vectors in V. If $m > d$ then S must span V.
(16) Let V be a subspace of $\mathbb{R}^n$ of dimension d and let S be a set of m vectors in V. If $m < d$ then S is linearly independent.
(17) Let V be a subspace of $\mathbb{R}^n$ of dimension d and let S be a set of m vectors in V. If $m = d$ and S spans V, then S is a basis for V.
(18) If A is an m x n matrix and $n > m$, then the null space of A is not $\{\mathbf{0}\}$.
(19) If A is an m x n matrix, then dim Null A + dim Col $A = n$.
(20) If A is an m x n and rank $A < n$, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

(21) The nullity of A always equals the nullity of A^T.

(22) The rank of A always equals the rank of A^T.

(23) If A is an m x n matrix, then dim Null A^T + dim Col $A = m$.

(24) If A is n x n and diagonalizable, then there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

(25) If A is an n x n matrix with n distinct real eigenvalues, then A is diagonalizable.

(26) If A is a square matrix with an eigenvalue λ_0 of multiplicity k, then the dimension of the eigenspace corresponding to λ_0 is also k.

Which of the following are subspaces of $\mathbb{R}^3$?

(1) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $y = x^2$.

(2) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x \geq 0$.

(3) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $z = 0$.

(4) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $3x - 2y - z = 0$.

(5) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x + y = 3$.

Which of the following are subspaces of M_{22}?

(1) The set of 2 x 2 symmetric matrices.

(2) The set of 2 x 2 matrices with zero trace.

(3) The set of 2 x 2 matrices with zero determinant.

(4) The set of 2 x 2 nonsingular matrices.

(5) The set of 2 x 2 diagonal matrices.

(6) The set of 2 x 2 lower triangular matrices.