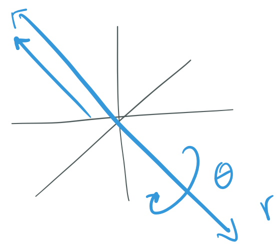
12.3. Flat three-manifolds

We now turn to flat three-manifolds. In dimension two, every orientation-preserving isometry of $\mathbb{R}^2$ is a translation, and this easily implies that every flat orientable closed surface is a torus. In dimension three we also have *rototranslations*, which produce more orientable manifolds.

Def: a rototranslation in $\mathbb{R}^3$ is a rotation of some angle θ along an axis r composed with a translation of some distance $t>0$ in the direction of r.

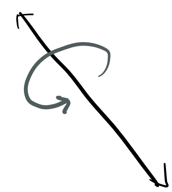


Our Goal:

Theorem 12.3.1. *A closed orientable 3-manifold M admits a flat metric if and only if it is a Seifert manifold with $e = \chi = 0$.*

12.3.1. Classification.

Every closed flat 3-manifold is isometric to $\mathbb{R}^3/_\Gamma$ for some $\underbrace{\text{crystallographic group}}_{\text{discrete subgroup w/ compact quotient}}$ $\Gamma < \mathrm{Isom}(\mathbb{R}^3)$ acting freely, see Section 4.4.4.



Every isometry of $\mathbb{R}^3$ can be written as $g(x) = Ax+b$ where $A \in O(3)$ and $b \in \mathbb{R}^3$ (generalizes to $\mathbb{R}^n$)

Exercise 12.3.2. Every element in Γ is either a translation or a rototranslation (defined in Example 4.4.7).

We rule out rotations since then the action of Γ on $\mathbb{R}^3$ wouldn't be free. Similarly if axis of rotation and $\vec{b}$ are not parallel, you will get a fixed pt.

We prove one half of Theorem 12.3.1.

Proposition 12.3.3. *Every closed orientable flat 3-manifold is a Seifert manifold with $e = \chi = 0$.*

Proof. We have $M = \mathbb{R}^3/_\Gamma$. Recall the exact sequence

$$0 \longrightarrow H \longrightarrow \Gamma \longrightarrow r(\Gamma) \longrightarrow 0$$

induced by $r : \mathrm{Isom}(\mathbb{R}^n) \to O(n)$, $Ax+b \mapsto Ax$

$0 \to \mathbb{R}^n \to \mathrm{Isom}(\mathbb{R}^n) \xrightarrow{r} O(n) \to 0$

where $H \triangleleft \Gamma$ is the translation subgroup and $r(\Gamma) < SO(3)$ is finite by Proposition 4.4.9. We now prove that Γ preserves a foliation of $\mathbb{R}^3$ into parallel lines that projects to a Seifert structure on M. ☆

black box for now

If $r(\Gamma)$ is trivial, then $\Gamma = H$ consists of translations and preserves many foliations into parallel lines that project to a Seifert structure on the quotient 3-torus M. If $r(\Gamma)$ is non-trivial, it is isomorphic to C_n, D_{2m}, T_{12}, O_{24}, or I_{60}. If $r(\Gamma) = C_n$ or D_{2m}, it has a common fixed vector line $l \subset \mathbb{R}^3$ and Γ preserves the foliation of lines parallel to l.

finite subgroups of $O(3)$

If $r(\Gamma) = T_{12}, O_{24}$, or I_{60} we obtain a contradiction as follows. In all cases we have $T_{12} \subset r(\Gamma)$. The group T_{12} consists of the identity, the π-rotations along the three coordinate axis, and the $\pm\frac{2\pi}{3}$-rotations along the axis spanned by $(1, 1, 1), (1, -1, -1), (-1, 1, -1)$, and $(1, -1, -1)$.

use the specific structure of this group to find a fixed pt for an element which contradicts free action of Γ on $\mathbb{R}^3$

Pick a rototranslation $h \in \Gamma$ with axis l parallel to $(1, 1, 1)$, and up to conjugating Γ by a translation we may suppose that l contains the origin $0 \in \mathbb{R}^3$. We have $h(0) = \underbrace{(d, d, d)}_{= b \text{ in } A\vec{x}+b}$ for some $d \neq 0$. Since h^3 is a translation we get $(3d, 3d, 3d) \in H$.

The group Γ and hence T_{12} acts on H via conjugation. Therefore H is T_{12}-symmetric and $(3d, -3d, -3d) \in H$ using the π-rotation along the first axis. Hence $t = (6d, 0, 0) = (3d, 3d, 3d) + (3d, -3d, -3d) \in H$. The composition $t \circ h^2$ has a fixed point, because it sends 0 to $(2d, 2d, 2d) - (6d, 0, 0) = (-4d, 2d, 2d)$ which is orthogonal to l: a contradiction.

In all cases M has a Seifert structure. By Bieberbach's Theorem (stated as Corollary 4.4.11) the manifold M is finitely covered by the 3-torus, and hence $\chi = e = 0$ by Proposition 10.3.26. □

Every crystallographic group has f.i. trans. subgroup $\cong \mathbb{Z}^n$.

$\Rightarrow$ Bieberbach: every closed flat n-mfd is covered by a flat torus.

we want from this:

M covered by 3-torus iff $\chi(S) = 0$ + $e = 0$

$$e(M) = \sum \frac{q_i}{p_i} = 0$$

$$\chi(B) = \chi(|B|) - \sum_i \left(1 - \frac{1}{p_i}\right) = 0$$

We now prove the other half.

Proposition 12.3.4. *Every closed Seifert manifold M with $e = \chi = 0$ admits a flat metric.*

Proof. There are six such Seifert manifolds up to diffeomorphism, listed in Table 10.3. We build a flat metric for each in Figure 12.2.

PAUSE: why only 6?

Proposition 10.3.38. *Every closed Seifert fibration with $\chi = 0$ and $e = 0$ is isomorphic to one of the seven listed in Table 10.3. These seven manifolds are all pairwise non-diffeomorphic, except*

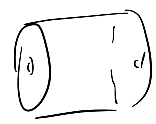
$$\big(S^2, (2,1), (2,1), (2,-1), (2,-1)\big) \cong K \mathbin{\tilde{\times}} S^1.$$

$\chi = 2 - \frac{1}{2} - \frac{2}{3} - \frac{5}{6}$

Proof. The closed orbifolds with $\chi = 0$ are

$$T,\ K,\ (\mathbb{RP}^2, 2, 2),\ (S^2, 2, 2, 2, 2),\ (S^2, 2, 3, 6),\ (S^2, 3, 3, 3),\ (S^2, 2, 4, 4).$$

It is easy to show that by imposing $e = 0$ we get the fibrations listed. The homology calculation is an easy exercise and luckily distinguishes all the manifolds except (of course) the two diffeomorphic ones (see Corollary 10.3.35). □



$K \tilde{\times} I$ $\quad$ $S^2(2,2)$

double

$K \tilde{\times} S^1$ $\quad$ $S^2(2,2,2,2$

M	$H_1(M, \mathbb{Z})$
$T \times S^1$	$\mathbb{Z}^3$
$K \mathbin{\tilde{\times}} S^1$	$\mathbb{Z} \times \mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}/_{2\mathbb{Z}}$
$\big(S^2, (2,1), (2,1), (2,-1), (2,-1)\big)$	$\mathbb{Z} \times \mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}/_{2\mathbb{Z}}$
$\big(S^2, (3,1), (3,1), (3,-2)\big)$	$\mathbb{Z} \times \mathbb{Z}/_{3\mathbb{Z}}$
$\big(S^2, (2,1), (4,-1), (4,-1)\big)$	$\mathbb{Z} \times \mathbb{Z}/_{2\mathbb{Z}}$
$\big(S^2, (2,1), (3,1), (6,-5)\big)$	$\mathbb{Z}$
$\big(\mathbb{RP}^2, (2,1), (2,-1)\big)$	$\mathbb{Z}/_{4\mathbb{Z}} \times \mathbb{Z}/_{4\mathbb{Z}}$

Table 10.3. The seven closed Seifert fibrations with $\chi = 0$ and $e = 0$. Two of these manifolds are actually diffeomorphic, so we get six closed Seifert manifolds up to diffeomorphism, distinguished by their homology.

Recall:

Corollary 10.3.35. *The following diffeomorphisms hold:*

$$\big(S^2, (2,1), (2,-1), (p,q)\big) \cong \big(\mathbb{RP}^2, (q,p)\big),$$
$$\big(S^2, (2,1), (2,1), (2,-1), (2,-1)\big) \cong K \mathbin{\tilde{\times}} S^1.$$

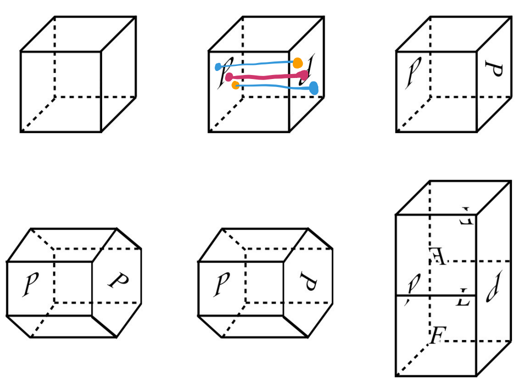


Figure 12.2. The six closed orientable flat 3-manifolds, up to diffeomorphism. Each is constructed by pairing isometrically the faces of a polyhedron in $\mathbb{R}^3$ according to the labels. When a face has no label, it is simply paired to its opposite by a translation. The polyhedra shown here are three cubes, two prisms with regular hexagonal basis, and one parallelepiped made of two cubes.

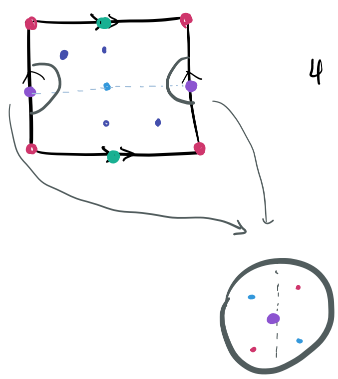
The figure shows six flat manifolds, constructed by identifying isometrically the faces of a polyhedron in $\mathbb{R}^3$. The reader is invited to check that each construction gives indeed a flat manifold, by verifying that the flat metric extends to the edges and to the vertices.

In all cases the foliation by parallel horizontal lines (orthogonal to the P faces) descends to a Seifert fibration on the flat manifold. By looking at these lines one checks that the base surface of the fibration is respectively

$$T,\ (S^2,2,2,2,2),\ (S^2,2,4,4),\ (S^2,2,3,6),\ (S^2,3,3,3),\ (\mathbb{RP}^2,2,2).$$

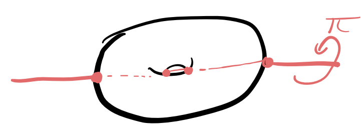
These orbifolds are obtained respectively from the figures by considering: the square torus, its quotient via a π-rotation, via a $\frac{\pi}{4}$-rotation, the quotient of a hexagon torus by a $\frac{\pi}{3}$-rotation, by a $\frac{2\pi}{3}$-rotation, and the quotient of a Klein bottle via a π-rotation.

These flat Seifert manifolds have $e = 0$ by Proposition 12.3.3, and hence they are precisely those listed in Table 10.3. □



4 cone pts

this is exactly T^2 mod out by hyperelliptic invol.



4 fixed pts are pts of intersection