The integer q is the *Euler number* of the circle bundle and is usually denoted with the letter e. We summarise our discovery:

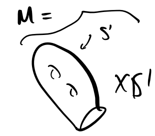
Note for ex we did above where $m' = 3 \cdot m + 2l$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$e = \frac{2}{3}$

Corollary 10.2.6. *For every $e \in \mathbb{Z}$ and every closed surface S there is a unique oriented circle bundle over S with Euler number e.*

A change of orientation for M transforms e into $-e$. Recall that every closed surface S has a trivial circle bundle $S \,\tilde{\times}\, S^1$ constructed by doubling the unique oriented line bundle on S.



Exercise 10.2.7. An oriented circle bundle over a closed surface is trivial $\iff e = 0 \iff$ the bundle has a section.

suppose $e = 0$ and did $(1,0)$ filling

We may see the Euler number of a bundle $M \to S$ over a closed S as an obstruction for the existence of a section.

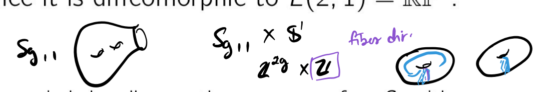
gives $\Rightarrow S \times S^1$

Remark 10.2.8. Every oriented n-dimensional vector bundle $E \to S$ over a closed oriented n-manifold S has a *Euler number* defined by taking two generic sections and counting their signed intersections. We briefly explain how this number is closely related to the one we defined here.

Each vector bundle $E \to S$ induces a sphere bundle $M \to S$: it suffices to fix a Riemannian metric on E and take the sub-bundle consisting of unit tangent vectors. When $n = 2$ we get a circle bundle $M \to S$ and the Euler number of $E \to S$ coincides with that of $M \to S$ that we defined above.

When E is the tangent bundle of S, the Euler number is the Euler characteristic $\chi(S)$. For instance, the unit tangent bundle of S^2 has Euler number $e = \chi(S^2) = 2$ and hence it is diffeomorphic to $L(2,1) = \mathbb{RP}^3$.

Section arg: Suppose I have M an S^1 bundle over closed S.



Exercise 10.2.9. Let M be a circle bundle over the genus-g surface S_g with Euler number e. We have $H_1(M, \mathbb{Z}) = \mathbb{Z}^{2g} \times \mathbb{Z}/_{e\mathbb{Z}}$.

$\pi : M \to S \supset S^1$

Corollary 10.2.10. *Let $M \to S_g$ and $M' \to S_{g'}$ be circle bundles with Euler numbers e and e'. The manifolds M and M' are diffeomorphic $\iff g = g'$ and $|e| = |e'|$.*

$\pi^{-1}(S^1)$ is S^1 bundle over S^1 in M

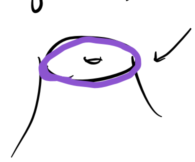
$\| \; S^1 \times S^1 = N$

Exercise 10.2.11. The circle bundle M over S^2 with Euler number e is diffeomorphic to the lens space $L(|e|, 1)$.

Hint. The base sphere S^2 decomposes into two discs, and the fibration over each disc is a solid torus. Therefore M is the union of two solid tori. □

M is obtained from N by doing D.F.

§ 10.3 Seifert Manifolds



10.3.1. Definition. We define the Seifert manifolds as Dehn fillings of trivial bundles over surfaces with boundary. Here are the details.

Let M be the (unique) oriented bundle $S \,\tilde{\times}\, S^1$ over a compact connected (possibly non-orientable) surface S with boundary. We denote by S the zero-section.

Let $T_1, \ldots, T_k$ be the boundary tori of M. On each T_i we choose an orientation for the meridian $m_i = T_i \cap \partial S$ and for the fibre l_i of the bundle so that the basis (m_i, l_i) for $H_1(T_i, \mathbb{Z})$ be positively oriented.

A (p_i, q_i)-Dehn filling on T_i kills the slope $p_i m_i + q_i l_i$. We say that the Dehn filling is *fibre-parallel* if $p_i = 0$, *i.e.* if it kills a fibre.

Definition 10.3.1. A *Seifert manifold* is any 3-manifold N obtained from M by Dehn filling some $h \leqslant k$ boundary tori in a non-fibre-parallel way, that is with $p_i \neq 0$ for all i.

The Seifert manifold is closed if $h = k$, and has $k - h$ boundary tori otherwise. It is not important to know which h tori are filled, in virtue of the following.

Proposition 10.3.2. *Every permutation of the boundary tori is realised by a self-diffeomorphism of M that preserves the pairs $\pm(m_i, l_i)$.*

Proof. Every permutation of the boundary circles of S is realised by a self-diffeomorphism of S, that extends orientation-preservingly to the orientable I-bundle and its double M. □

The pair (p_i, q_i) is determined up to sign, so we can always suppose $p_i > 0$ and we fully encode the Seifert manifold N using the following notation:

$$N = \big(\hat{S}, (p_1, q_1), \ldots, (p_h, q_h)\big) \tag{6}$$

where $\hat{S}$ is S with h boundary components capped. The reason for using $\hat{S}$ instead of S is that N has a particular fibration onto $\hat{S}$, as we will soon see. Before constructing this fibration we list some simple examples that should hopefully help the reader to familiarise with the notation (6), that will be used extensively in the whole chapter.

Example 10.3.3. The Seifert manifold $\big(S_g, (1, e)\big)$ is the circle bundle over the orientable genus-g surface S_g with Euler number e, by construction. In particular $\big(S_g, (1, 0)\big) = S_g \times S^1$.

Example 10.3.4. The Seifert manifold $\big(S^2, (p, q)\big)$ is diffeomorphic to the lens space $L(q, p)$.

The following facts follow from Exercise 10.1.7.

Exercise 10.3.5. The Seifert manifold $\big(D^2, (p, q)\big)$ is a solid torus.