§11.5 The JSJ Decomposition !!

Let $S = T_1 \sqcup \cdots \sqcup T_k$

be a set of disjoint essential tori $T_i \subset \text{int}(M)$. We say that S is a *torus decomposition* of M if it decomposes M into blocks that are either:

- torus (semi-)bundles,
- Seifert manifolds, or
- simple manifolds.

A torus decomposition is *minimal* if no proper subset of S is a torus decomposition. We prove here the following.

Theorem 11.5.1 (JSJ decomposition). *Let M be an orientable irreducible and ∂-irreducible compact 3-manifold with (possibly empty) boundary consisting of tori. A minimal torus decomposition for M exists and is unique up to isotopy.*

Such a minimal decomposition is called the *canonical torus decomposition* or the *JSJ decomposition* of M. The canonical torus decomposition may be empty: this holds precisely when M is itself a torus (semi-)bundle, Seifert, or simple.

Remark 11.5.2. Torus (semi-)bundles are closed: therefore if M is not itself a torus (semi-)bundle, the blocks of its canonical decomposition are either Seifert or simple.

11.5.2. Existence and uniqueness. Let M be an orientable irreducible and ∂-irreducible compact 3-manifold with (possibly empty) boundary consisting of tori. We now prove Theorem 11.5.1. Let us start by showing existence.

Proposition 11.5.3. *The manifold M has a torus decomposition.*

Proof. Let $T_1, \ldots, T_k$ be a maximal set of disjoint non-parallel essential tori in M, which exists by Corollary 9.4.8. We now prove that $S = T_1 \sqcup \cdots \sqcup T_k$ is a torus decomposition.

Suppose it is not: one block N of the decomposition is neither a (semi-)bundle, nor Seifert, nor simple. The block N is irreducible and ∂-irreducible since these properties are preserved after cutting along incompressible surfaces. Being not simple, it contains an essential annulus A or an essential torus T. irred. + ∂ irred rule out S^2, D.

In the latter case we can add T to the family $T_1, \ldots, T_k$ and get a contradiction since S is maximal. In the former case Lemma 11.2.10 applies and N is Seifert. $\square$ → black box

Since M has a torus decomposition, it certainly has a minimal one. We now prove that it is unique.

Proposition 11.5.4. *The manifold M has a unique minimal torus decomposition up to isotopy.*

Proof. Let $S = T_1 \sqcup \cdots \sqcup T_k$ and $S' = T'_1 \sqcup \cdots \sqcup T'_{k'}$ be two minimal torus decompositions for M. We minimise their transverse intersections, so that $S \cap S'$ consists of essential circles cutting some tori into annuli.

Let T'_i be decomposed into some annuli. Each such annulus is essential in $M \setminus S$, hence it is contained in some non-simple block, *i.e.* a Seifert one. It is contained there horizontally or vertically: in the former case, the block is $(D, (2,1), (2,1))$, $S_* \tilde{\times} S^1$, or $A \times S^1$ with S_* the Möbius strip. The first two blocks are diffeomorphic, and by swapping the fibration the annulus becomes vertical. The third block $T \times I$ is excluded since S is minimal.

rules out horiz.

10.4.2. Horizontal and vertical surfaces. We now want to study how Seifert manifolds may contain interesting surfaces. Let $M \to S$ be a Seifert fibration. A properly embedded surface $\Sigma \subset M$ is

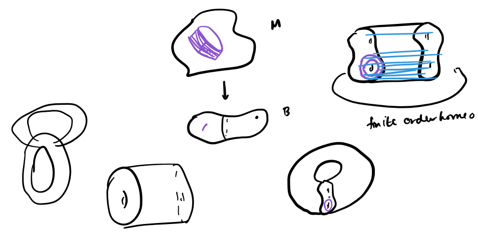
- *vertical* if it is a union of some regular fibres,
- *horizontal* if it is transverse to all fibres.

If Σ is vertical, it is either an annulus, a torus, or a Klein bottle, projecting respectively to an arc, an orientation-preserving, or an orientation-reversing simple closed curve that avoids the cone points. Vertical surfaces are in 1-1 correspondence with 1-dimensional objects in S and are thus easily determined.

Now all annuli in T'_i are vertical. Two consequent vertical annuli are separated by some torus T_j; since the two annuli are fibered, the fibers of the two Seifert blocks incident to T_j are isotopic: hence the two blocks glue to a bigger Seifert block and T_j can be removed, a contradiction since S is minimal.

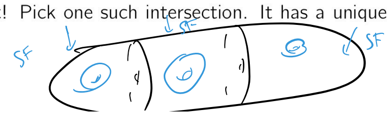
rules out vert

⇒ We have shown that $S \cap S' = \varnothing$. If T_i is parallel to T'_j we superpose the two tori, cut M along $T_i = T'_j$ and proceed by induction. Now we suppose by contradiction that there is no parallelism.



Every T'_i is an essential vertical torus in a Seifert block of $M \setminus S$, and vice versa. This easily implies that all the blocks in $M \setminus S$, $M \setminus S'$, and all their intersections are Seifert! Pick one such intersection. It has a unique Seifert

break into SF blocks



that glue up to one bis SF block $\Rightarrow \Sigma$ bmin. of S and S'

fibration, unless it is $K \tilde{\times} I$ which may fiber in two ways. Since $\partial(K \tilde{\times} I)$ is connected, one block is $K \tilde{\times} I$ itself and we change the fibration on this block if necessary. Now all intersections and all blocks have unique fibrations and they all glue to a Seifert fibration for M, a contradiction. □

The tori decomp. was not minimal.

The proof of Theorem 11.5.1 is complete.

Remark 11.5.5. The sphere decomposition of Theorem 9.2.29 and the torus decomposition of Theorem 11.5.1 differ in two aspects: (i) the set of decomposing spheres is *not* canonical up to isotopy, while the set of tori is; (ii) on the other hand, after cutting along the spheres and capping off we get a canonical set of prime manifolds, whereas if we cut along the tori we get some canonical manifolds with toric boundaries, but there is no canonical way to cap them off.

Chapter 12 + extra

Suppose we have proven that M has $\chi^{orb} > 0$ and $e \neq 0$ iff its a spherical mfd.

Let's try to understand these mfds and their SFS better.

$\chi(S^2) - \Sigma(1 - \frac{1}{p_i})$

9.2 Spherical manifolds according to their base orbrifolds

We now further describe the structure of spherical 3-manifolds according on their base orbifolds. Since $\chi^{orb} B > 0$ there are three infinite families for B given by $S^2(n,n), S^2(2,2,n)$ and $\mathbb{R}\mathrm{P}^2(n)$ with $n > 0$. The remaining possibilities for B are $S^2(2,3,n)$ with $n = 3,4$ or 5.

obtained from S^2 via quotient by a finite group of Isometries (finite subgroup of $O(3)$ - rotations about origin)

The two families with base orbifold $S^2(2,2,n)$ and $\mathbb{R}\mathrm{P}^2(n)$ all have a Seifert fibering with base $S^2(2,2,n)$, see Hatcher (2.3)(d). Additionally, $S^2(n,n)$ is an orbifold cover of $S^2(2,2,n)$ of degree 2. If $B = S^2(n,n)$ then M is a *lens space* (defined below) add a reference here. Therefore, all manifolds M' with base $S^2(2,2,n)$ are *prism manifolds*, i.e. they have a two-fold covering that is a lens space M, induced by the orbifold cover. In

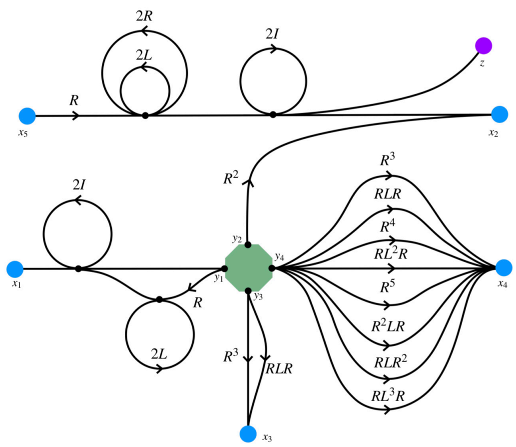
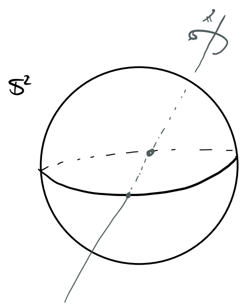
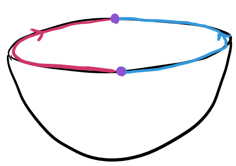


Figure 7: Branched manifold for Spherical Geometry

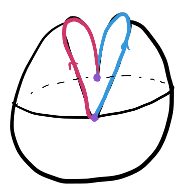


/ rotate by π

$\cong$

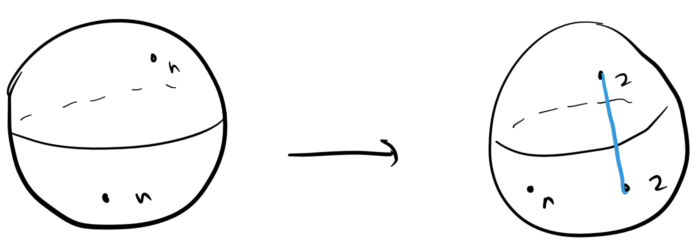


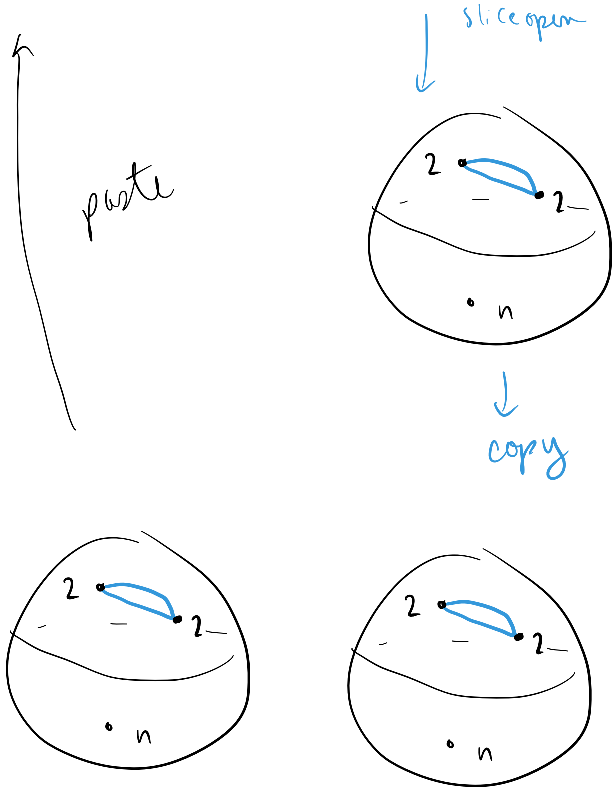
$=$



$= S^2(2,2)$

Same but rotation by $\frac{2\pi}{3}$ gives $S^2(3,3)$





$$\begin{tikzcd} M \arrow[r] \arrow[d] & S^2(n,n) \arrow[d] \\ M' \arrow[r] & S^2(2,2,n) \end{tikzcd}$$

S^2 

$\times S^1$

Cut out two solid tori