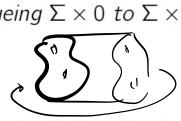
Chapter 11:

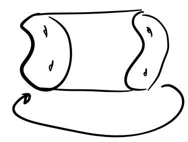
Recall that a mfd is simple if it contains no essential sphere, ess. disc, ess. torus, and ess. annulus.

11.4.1. Surface bundles. A *surface bundle over S^1* is a fibre bundle $M \to S^1$ of a compact orientable 3-manifold M (possibly with boundary) over the circle, whose fibre Σ is a connected compact orientable surface. If M has boundary, then Σ also has, and ∂M consists of tori fibering over S^1.

Proposition 11.4.1. *Every surface bundle over S^1 is constructed by taking $\Sigma \times [0,1]$ and glueing $\Sigma \times 0$ to $\Sigma \times 1$ via an orientation-preserving diffeomorphism ψ.*



Proof. One such glueing clearly gives rise to a surface bundle over S^1. Conversely, by cutting a surface bundle over S^1 along a fibre we get a surface bundle over the interval, which is a product $\Sigma \times [0,1]$. ✓ □



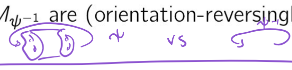
The diffeomorphism ψ is the *monodromy* of the surface bundle M_ψ. Since isotopic glueings produce diffeomorphic manifolds, the three-manifold M_ψ depends only of the class of ψ in the mapping class group $\mathrm{MCG}(\Sigma)$ on Σ. More than that, it actually depends only on its conjugacy class: $\cong \mathrm{Diffeo}^+(\Sigma)/\sim$

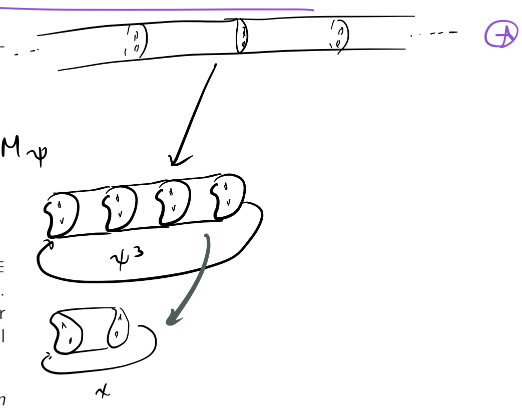
→ Proposition 11.4.2. *If ψ and ϕ are conjugate in $\mathrm{MCG}(\Sigma)$, then M_ψ and M_ϕ are diffeomorphic.*

$g \psi g^{-1} = \varphi$

Proof. The diffeomorphism $g\colon \Sigma \to \Sigma$ that conjugates them extends to $\Sigma \times [0,1]$ and gives a diffeomorphism $M_\psi \to M_\phi$. □

The manifolds M_ψ and $M_{\psi^{-1}}$ are (orientation-reversingly) diffeomorphic.





11.4.2. Properties. We now start to investigate the topological properties of surface bundles. Let $M \to S^1$ be a surface bundle with fibre Σ.

Exercise 11.4.3. The maps $\Sigma \to M \to S^1$ induce an exact sequence

$$0 \longrightarrow \pi_1(\Sigma) \longrightarrow \pi_1(M) \longrightarrow \pi_1(S^1) \longrightarrow 0.$$

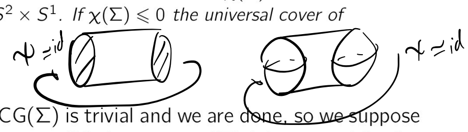
This implies in particular that $\pi_1(M)$ surjects onto $\mathbb{Z}$ and therefore:

Corollary 11.4.4. *We have $b_1(M) \geqslant 1$.*

In other terms, the surface fibre Σ is non-separating and hence $[\Sigma] \in H_2(M, \partial M)$ is non-trivial (and has infinite order, since there is no torsion there).

Note that there is an obvious degree-n regular covering $M_{\psi^n} \to M_\psi$ for every n and an infinite regular covering $\Sigma \times \mathbb{R} \to M_\psi$ induced by the normal subgroup $\pi_1(\Sigma) \triangleleft \pi_1(M)$.

Proposition 11.4.5. *The fibre Σ is an essential surface. If $\chi(\Sigma) > 0$ then M is diffeomorphic to $D \times S^1$ or $S^2 \times S^1$. If $\chi(\Sigma) \leqslant 0$ the universal cover of* $\mathrm{int}(M)$ *is $\mathbb{R}^3$ and M is Haken.*



Proof. If $\chi(\Sigma) > 0$ then $\mathrm{MCG}(\Sigma)$ is trivial and we are done, so we suppose $\chi(\Sigma) \leqslant 0$. The fibre Σ is incompressible because $\pi_1(\Sigma)$ injects, and is also ∂-incompressible by a doubling argument (the double DM fibres to S^1 with incompressible fibre $D\Sigma$, hence Σ is ∂-incompressible). The fibre is clearly not ∂-parallel, hence it is essential.

The manifold M is covered by $\Sigma \times \mathbb{R}$, whose interior is covered by $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$: hence M is irreducible. It is also ∂-irreducible because its double also fibres and hence is irreducible. Therefore M is Haken. □

no ess. spheres

no essential discs

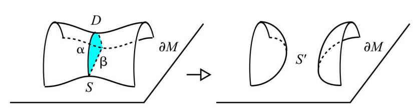
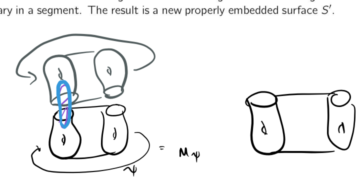
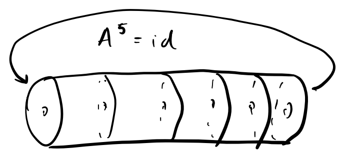


Figure 9.9. We can also surger a surface S along a disc D touching the boundary in a segment. The result is a new properly embedded surface S'.



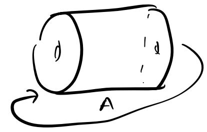
11.4.5. Torus bundles. A *torus bundle* is of course a surface bundle $M \to S^1$ with fibre a torus T. We fix a basis for $\pi_1(T)$, so that $\mathrm{MCG}(T) = \mathrm{SL}_2(\mathbb{Z})$. By what said above, every matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ defines a torus bundle M_A with monodromy A. We want to understand when M_A is a Seifert manifold.



$M_{A^5} = T^2 \times S^1$
$\Downarrow$
$e=0$
$\chi(T^2)=0$

5-fold reg. cover
$A^5 = id$

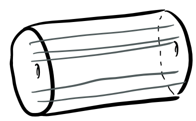
Exercise 11.4.13. A matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ has finite order if and only if $A = \pm I$ or $|\mathrm{tr}A| < 2$. ~~Every finite order A is conjugate to one of~~



$\Rightarrow e=0$
$\chi = 0$

Proposition 11.4.14. *Let $M = M_A$ be a torus bundle with monodromy $A \neq \pm I$. The following holds:*

- *if $|\mathrm{tr}A| < 2$ then M is a Seifert manifold with $e = 0$ and $\chi = 0$,*
- *if $|\mathrm{tr}A| = 2$ then M is a Seifert manifold with $e \neq 0$ and $\chi = 0$,*
- *if $|\mathrm{tr}A| > 2$ then M is not a Seifert manifold.*



Proof. Consider $T \times [-1, 1]$ foliated by lines $\{x\} \times [-1, 1]$. The foliation extends to M_A. If A has finite order, then M_A is finitely covered by $M_I = T \times S^1$ and hence all fibers are compact. Therefore M_A is Seifert fibered and covered by $T \times S^1$, and we get $e = \chi = 0$.

If $|\mathrm{tr}A| = 2$ then A is conjugate to $\begin{pmatrix} \pm1 & e \\ 0 & \pm1 \end{pmatrix}$ and we use Exercise 11.4.11. Proposition 11.4.10 easily shows that all the Seifert manifolds that are torus bundles are realised with $|\mathrm{tr}A| \leqslant 2$, hence if $|\mathrm{tr}A| > 2$ the manifold M is not Seifert by Proposition 11.4.12. $\square$

Exercise 11.4.11. For every $e \in \mathbb{Z}$ there are diffeomorphisms
$$M_{\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}} \cong (T, (1, e)), \quad M_{\begin{pmatrix} -1 & e \\ 0 & -1 \end{pmatrix}} \cong (K, (1, e)).$$

When $|\mathrm{tr}A| > 2$ we say that the monodromy A is *Anosov*.

These are the Sol mfds.

11.4.6. Bundles with $\chi(\Sigma) < 0$. Proposition 11.4.12 does not extend to surface bundles with $\chi(\Sigma) < 0$; indeed it may happen that non-conjugate monodromies in $\mathrm{MCG}(\Sigma)$ give rise to diffeomorphic manifolds and understanding when this happens is a hard problem.

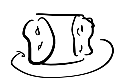
Proposition 11.4.14 extends nevertheless and reflects the trichotomy of mapping classes. Let Σ be a closed orientable surface with $\chi(\Sigma) < 0$. Recall from Section 8.4 that every element $\psi \in \mathrm{MCG}(\Sigma)$ is either finite order, reducible, or pseudo-Anosov.

f.o. $\psi^k = id$ some k
red $\exists \gamma_1, \ldots, \gamma_n$, $\gamma = \Gamma$
s.t. Γ is inv. under ψ
pA

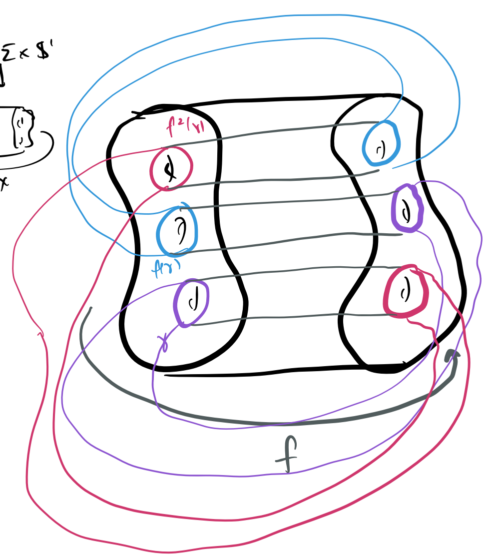
Proposition 11.4.15. *Let M_ψ be a surface bundle with fibre Σ and monodromy $\psi \in \mathrm{MCG}(\Sigma)$. The following holds:*

- *if ψ has finite order, then M_ψ is Seifert with $\chi < 0$ and $e = 0$,*
- *if ψ is reducible, then M_ψ contains an essential torus,*
- *if ψ is pseudo-Anosov, then M_ψ is simple and not Seifert.*

$\Sigma \times S^1$
$\downarrow$



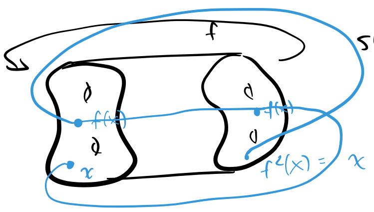
χ



Proof. Same proof as Proposition 11.4.14. If ψ has finite order, it is an isometry for some hyperbolic metric on Σ, and the line fibration of $\Sigma \times [-1, 1]$ glues to a Seifert fibration for M_ψ.

with $\psi(\gamma_i) = \gamma_{i+1}$ cyclically; by gluing the annuli $\gamma_i \times [-1, 1]$ we get a torus $T \subset M_\psi$. It is essential because by cutting along it we still get a fibration over S^1 with fibers having $\chi \leqslant 0$.

If ψ is pseudo-Anosov there are no essential tori $T \subset M_\psi$, for by minimising $T \cap \Sigma$ then $M_\psi \setminus T$ would consist of essential annuli of type $\gamma \times [-1, 1]$ and hence ψ would be reducible. The manifold M_ψ is not Seifert because the fibre Σ would become a horizontal surface: then M would be covered by $\Sigma \times S^1$ and hence ψ would be of finite order. $\square$



suppose that $f^2 = id$

§11.5 The JSJ Decomposition !!

Let $S = T_1 \sqcup \cdots \sqcup T_k$

be a set of disjoint essential tori $T_i \subset \text{int}(M)$. We say that S is a *torus decomposition* of M if it decomposes M into blocks that are either:

- torus (semi-)bundles,
- Seifert manifolds, or
- simple manifolds.

A torus decomposition is *minimal* if no proper subset of S is a torus decomposition. We prove here the following.

Theorem 11.5.1 (JSJ decomposition). *Let M be an orientable irreducible and ∂-irreducible compact 3-manifold with (possibly empty) boundary consisting of tori. A minimal torus decomposition for M exists and is unique up to isotopy.*

Such a minimal decomposition is called the *canonical torus decomposition* or the *JSJ decomposition* of M. The canonical torus decomposition may be empty: this holds precisely when M is itself a torus (semi-)bundle, Seifert, or simple.

Remark 11.5.2. Torus (semi-)bundles are closed: therefore if M is not itself a torus (semi-)bundle, the blocks of its canonical decomposition are either Seifert or simple.

11.5.2. Existence and uniqueness. Let M be an orientable irreducible and ∂-irreducible compact 3-manifold with (possibly empty) boundary consisting of tori. We now prove Theorem 11.5.1. Let us start by showing existence.

Proposition 11.5.3. *The manifold M has a torus decomposition.*

Proof. Let $T_1, \ldots, T_k$ be a maximal set of disjoint non-parallel essential tori in M, which exists by Corollary 9.4.8. We now prove that $S = T_1 \sqcup \cdots \sqcup T_k$ is a torus decomposition.

Suppose it is not: one block N of the decomposition is neither a (semi-)bundle, nor Seifert, nor simple. The block N is irreducible and ∂-irreducible since these properties are preserved after cutting along incompressible surfaces. Being not simple, it contains an essential annulus A or an essential torus T. irred. + ∂ irred rule out S^2, D.

In the latter case we can add T to the family $T_1, \ldots, T_k$ and get a contradiction since S is maximal. In the former case Lemma 11.2.10 applies and N is Seifert. $\square$

black box

Since M has a torus decomposition, it certainly has a minimal one. We now prove that it is unique.

Proposition 11.5.4. *The manifold M has a unique minimal torus decomposition up to isotopy.*

Proof. Let $S = T_1 \sqcup \cdots \sqcup T_k$ and $S' = T'_1 \sqcup \cdots \sqcup T'_{k'}$ be two minimal torus decompositions for M. We minimise their transverse intersections, so that $S \cap S'$ consists of essential circles cutting some tori into annuli.

Let T'_i be decomposed into some annuli. Each such annulus is essential in $M \setminus S$, hence it is contained in some non-simple block, *i.e.* a Seifert one. It is contained there horizontally or vertically: in the former case, the block is $(D, (2,1), (2,1))$, $S_* \tilde{\times} S^1$, or $A \times S^1$ with S_* the Möbius strip. The first two blocks are diffeomorphic, and by swapping the fibration the annulus becomes vertical. The third block $T \times I$ is excluded since S is minimal.

rules out horiz.

10.4.2. Horizontal and vertical surfaces. We now want to study how Seifert manifolds may contain interesting surfaces. Let $M \to S$ be a Seifert fibration. A properly embedded surface $\Sigma \subset M$ is

- *vertical* if it is a union of some regular fibres,
- *horizontal* if it is transverse to all fibres.

If Σ is vertical, it is either an annulus, a torus, or a Klein bottle, projecting respectively to an arc, an orientation-preserving, or an orientation-reversing simple closed curve that avoids the cone points. Vertical surfaces are in 1-1 correspondence with 1-dimensional objects in S and are thus easily determined.

Now all annuli in T'_i are vertical. Two consequent vertical annuli are separated by some torus T_j; since the two annuli are fibered, the fibers of the two Seifert blocks incident to T_j are isotopic: hence the two blocks glue to a bigger Seifert block and T_j can be removed, a contradiction since S is minimal.

rules out vert

$\Longrightarrow$ We have shown that $S \cap S' = \varnothing$. If T_i is parallel to T'_j we superpose the two tori, cut M along $T_i = T'_j$ and proceed by induction. Now we suppose by contradiction that there is no parallelism.

Every T'_i is an essential vertical torus in a Seifert block of $M \setminus S$, and vice versa. This easily implies that all the blocks in $M \setminus S$, $M \setminus S'$, and all their intersections are Seifert! Pick one such intersection. It has a unique Seifert fibration, unless it is $K \tilde{\times} I$ which may fiber in two ways. Since $\partial(K \tilde{\times} I)$ is connected, one block is $K \tilde{\times} I$ itself and we change the fibration on this block if necessary. Now all intersections and all blocks have unique fibrations and they all glue to a Seifert fibration for M, a contradiction. □

The tori decomp. was not minimal.

The proof of Theorem 11.5.1 is complete.

Remark 11.5.5. The sphere decomposition of Theorem 9.2.29 and the torus decomposition of Theorem 11.5.1 differ in two aspects: (i) the set of decomposing spheres is *not* canonical up to isotopy, while the set of tori is; (ii) on the other hand, after cutting along the spheres and capping off we get a canonical set of prime manifolds, whereas if we cut along the tori we get some canonical manifolds with toric boundaries, but there is no canonical way to cap them off.