10.3.2. Seifert fibrations. As we anticipated, a Seifert manifold N as in (6) possesses some kind of singular fibration over the filled surface $\hat{S}$. We clarify this point here by defining the notion of *Seifert fibration*.

Let (p, q) be two coprime integers with $p > 0$. A *standard fibered solid torus* with coefficients (p, q) is the solid torus

$$D \times [0, 1]/_{\psi}$$

where $\psi\colon D \times 0 \to D \times 1$ is a rotation of angle $2\pi\frac{q}{p}$. The fibration into vertical segments $\{\mathrm{pt}\} \times [0, 1]$ extends to a fibration into circles of the solid torus. The

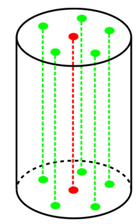
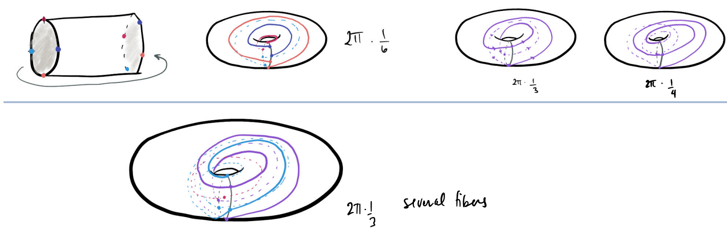
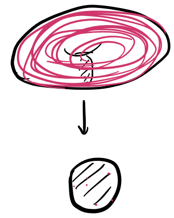


Figure 10.10. A standard fibered solid torus. We identify the top and bottom discs by a $2\pi\frac{q}{p}$ rotation, for some q coprime with p. Here $p = 5$. Every non-central fibre (green) winds p times along the central fibre (red).



central fibre obtained by identifying the endpoints of $0 \times [0, 1]$ is the core of the solid torus, and every non-central fibre winds p times around the core of M: see Figure 10.10.

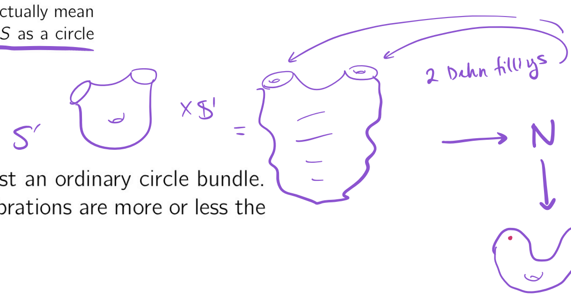
The positive number p is the *multiplicity* of the central fibre. If $p = 1$ the fibered solid torus is diffeomorphic to the usual product fibration $D \times S^1$ and the central fibre is *regular*. If $p > 1$ the central fibre is *singular*.



Definition 10.3.7. A *Seifert fibration* is a partition of a compact oriented 3-manifold N with (possibly empty) boundary into circles, such that every circle has a fibered neighbourhood diffeomorphic to a standard fibered solid torus.

The map $N \to S$ is in fact what we call a Seifert fibration. The surface S may have boundary and may be non-orientable, and its interior has a natural orbifold structure: if the preimage of $x \in S$ is a fibre of order p, we see x as a cone point of order p, see Section 6.2.4. If N has boundary, then S also has, and we say that S itself is an orbifold for simplicity although we actually mean only its interior. Morally, we should consider the fibration $N \to S$ as a circle bundle over the orbifold S.

S is now an orbifold



A Seifert fibration without singular fibres is just an ordinary circle bundle. We now show that Seifert manifolds and Seifert fibrations are more or less the same thing.

Proposition 10.3.9. *The Seifert manifold*

$$N = (S, (p_1, q_1), \dots, (p_h, q_h))$$

has a Seifert fibration $N \to S$ over the orbifold

$$(S, p_1, \dots, p_h).$$

Every Seifert fibration arises in this way.

Be careful the bundle is actually $N \to B = (S, p_1, \dots, p_h)$

$|B|$

	$\chi<0$	$\chi>0$	$\chi=0$
$e=0$	$\mathbb{H}^2\times\mathbb{R}$	$S^2\times\mathbb{R}$	$\mathbb{R}^3$
$e\neq0$	$\widetilde{SL_2}$	S^3	Nil

Goal 1 for today!!!

Proposition 10.3.26. *A closed Seifert fibration $M \to S$ has:*

- $\chi(S) > 0$ *and* $e = 0 \iff M$ *is covered by* $S^2 \times S^1$,
- $\chi(S) > 0$ *and* $e \neq 0 \iff M$ *is covered by* S^3,
- $\chi(S) = 0$ *and* $e = 0 \iff M$ *is covered by the 3-torus,*
- $\chi(S) = 0$ *and* $e \neq 0 \iff M$ *is covered by a twisted bundle over* T,
- $\chi(S) < 0$ *and* $e = 0 \iff M$ *is covered by* $S_g \times S^1$ *for some* $g > 1$,
- $\chi(S) < 0$ *and* $e \neq 0 \iff M$ *is covered by a twisted bundle over* S_g *for some* $g > 1$.

Def: Orbifold euler characteristic $\chi^{orb}(B) = \chi(|B|) - \sum_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$

ex: $(S_2, 2, 3) = B \quad \chi^{orb}(B) = -2 - \frac{1}{2} - \frac{2}{3} < 0$

$(S^2, 3, 4) = B \quad \chi^{orb}(B) = 2 - \frac{2}{3} - \frac{3}{4} > 0$

$(S^2, 3, 4, 57) = B \quad \chi^{orb}(B) = 2 - \frac{2}{3} - \frac{3}{4} - \frac{56}{57} < 0$

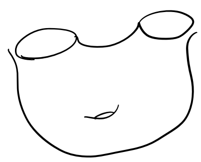
We have seen that the notation

$$N = \big(S, (p_1, q_1), \ldots, (p_h, q_h)\big) \tag{7}$$

defines a Seifert fibration $N \to S$ and a Seifert manifold N.

10.3.4. Euler number. We now extend the notion of Euler number from ordinary to Seifert fibrations. We define the *Euler number* of the fibration (7) to be the rational number

$$e(N) = \sum_{i=1}^{h} \frac{q_i}{p_i}.$$



$\times S^1 \xrightarrow[\text{of slope}]{\text{2 Dehn Fills}} N$

$q_1 = 1$
$p_1 = 3$
$q_2 = 2$
$p_2 = 5$

$e(N) = \frac{1}{3} + \frac{2}{5}$

Like ordinary fibrations, Seifert fibrations behave well with respect to coverings. Let $M \to S$ be a Seifert fibration and $\tilde{M} \to M$ a covering. The foliation into circles of M lifts to a foliation into circles or lines in $\tilde{M}$, with some quotient space $\tilde{S}$.

Proposition 10.3.17. *The quotient $\tilde{S}$ is an orbifold covering of S.*

- *If $\tilde{M}$ foliates in circles then $\tilde{M} \to \tilde{S}$ is a Seifert fibration,*
- *If $\tilde{M}$ foliates in lines then $\tilde{M} \to \tilde{S}$ is a line bundle.*

In the second case $\tilde{S}$ has no singular points.

10.3.7. Finite-degree coverings. We define a *finite-degree covering* of a Seifert fibration $M \to S$ to be a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \tilde{S} & \longrightarrow & S \end{array}$$

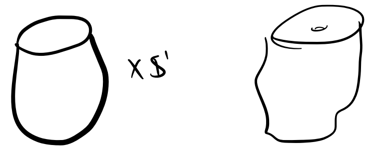
where $\tilde{M} \to \tilde{S}$ is a Seifert fibration, $\tilde{M} \to M$ is a finite-degree covering, and $\tilde{S} \to S$ is an orbifold covering. ~~Proposition 10.3.17 implies the following~~.

[redacted] that an orbifold is *very good* when it is finitely covered by a surface. Every locally orientable 2-orbifold is very good except the bad orbifolds $S^2(p_1)$ and $S^2(p_1, p_2)$ with $p_1 \neq p_2$, see Theorem 6.2.10 and Corollary 6.2.11.

Corollary 10.3.20. *If S is very good, every Seifert fibration $M \to S$ is finitely covered by a circle bundle over a surface.*

Proof. Pull-back the fibration along the surface cover $\tilde{S} \to S$. $\square$

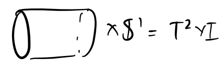
Ready for proof of :

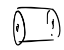


Proposition 10.3.26. *A closed Seifert fibration $M \to S$ has:*

- $\chi(S) > 0$ *and* $e = 0 \iff M$ *is covered by* $S^2 \times S^1$,
- $\chi(S) > 0$ *and* $e \neq 0 \iff M$ *is covered by* S^3,
- $\chi(S) = 0$ *and* $e = 0 \iff M$ *is covered by the 3-torus,*
- $\chi(S) = 0$ *and* $e \neq 0 \iff M$ *is covered by a twisted bundle over* T,
- $\chi(S) < 0$ *and* $e = 0 \iff M$ *is covered by* $S_g \times S^1$ *for some* $g > 1$,
- $\chi(S) < 0$ *and* $e \neq 0 \iff M$ *is covered by a twisted bundle over* S_g *for some* $g > 1$.

$D \times S^1$ = solid torus





Proof. If S is a bad orbifold, then $S = S^2(p)$ or $S^2(p_1, p_2)$ with $p_1 \neq p_2$ and hence we get $e \neq 0$ and $\chi(S) > 0$. The manifold M is a lens space by Example 10.3.10 and is hence covered by S^3.

If S is good, the fibration $M \to S$ is covered by a circle bundle $\tilde{M} \to \tilde{S}$ over an orientable closed surface $\tilde{S}$ by Corollary 10.3.20. By Proposition 10.3.22 the numbers $\chi(\tilde{S})$ and $e(\tilde{M})$ have the same signs of $\chi(S)$ and $e(M)$. Therefore $\tilde{S} = S^2, T$, or S_g with $g > 1$, depending on whether $\chi(S)$ is positive, null, or negative. Exercise 10.2.7 says that the circle bundle $\tilde{M} \to \tilde{S}$ is trivial $\iff e(\tilde{M}) = 0 \iff e(M) = 0$. Note that a non-trivial bundle over S^2 is a lens space $L(e, 1)$ with $e \neq 0$ and is hence covered by S^3.

$$\begin{tikzcd} \tilde{M} \arrow[r] \arrow[d] & M \arrow[d] \\ \tilde{S} \arrow[r] & S \end{tikzcd}$$

$$\begin{tikzcd} & \tilde{M} \text{ univ.} \arrow[dl] \arrow[dr] & \\ M_1 \arrow[dr] & & M_2 \arrow[dl] \\ & M & \end{tikzcd}$$

We have proved that M is covered (according to the signs of χ and e) by a manifold belonging to one of the six types: S^3, $S^2 \times S^1$, the 3-torus, a twisted bundle over T, $S_g \times S^1$, and a twisted bundle over S_g. It remains to prove that M cannot be covered by two manifolds M_1, M_2 belonging to two different types: this holds because manifolds of distinct types are not commensurable. To prove that, note that the finite cover of a manifold of one of the six types is a manifold of the same type, and Corollary 10.2.10 implies that a manifold cannot belong to two different types. If two manifolds of distinct types were commensurable they would be covered by a manifold belonging to both types, yielding a contradiction. □

Corollary 10.3.29. *A closed Seifert fibration $M \to S$ has $e = 0 \iff$ it is finitely covered by a trivial circle bundle.*

Goal 2: Classify SF

10.3.3. Classification of Seifert fibrations. We say that two Seifert fibrations $\pi_1 \colon N_1 \to S$, $\pi_2 \colon N_2 \to S$ are *isomorphic* if there is a diffeomorphism $\psi \colon N_1 \to N_2$ such that $\pi_1 = \pi_2 \circ \psi$. Two different notations as in (7) may describe isomorphic fibrations, but this phenomenon is completely understood.

$$\begin{tikzcd} N_1 \arrow[rr, "\psi"] \arrow[dr, "\pi_1"'] & \arrow[d, phantom, "\circlearrowright"] & N_2 \arrow[dl, "\pi_2"] \\ & S & \end{tikzcd}$$

Proposition 10.3.11. *Two notations as in (7) describe two orientation-preservingly isomorphic Seifert fibrations if and only if they are related by a finite sequence of the following moves and their inverses:*

(8) $\qquad (p_i, q_i), (p_{i+1}, q_{i+1}) \longmapsto (p_i, q_i + p_i), (p_{i+1}, q_{i+1} - p_{i+1}),$

(9) $\qquad (p_1, q_1), \ldots, (p_h, q_h) \longmapsto (p_1, q_1), \ldots, (p_h, q_h), (1, 0),$

(10) $\qquad (p_i, q_i) \longmapsto (p_i, q_i + p_i) \quad$ if $\partial N \neq \varnothing$,

and permutations of the pairs (p_i, q_i)'s.

$\frac{3}{2} \to \frac{5}{2}$ $\quad$ $\frac{1}{3} \mapsto \frac{-2}{3}$

ex. $(2,3), (3,1) \longmapsto (2,5), (3,-2)$

think "untwist in one solid torus and retwist in another"

32!

drill + then trivially fill

10.3.4. Euler number. We now extend the notion of Euler number from ordinary to Seifert fibrations. We define the *Euler number* of the fibration (7) to be the rational number

$$e(N) = \sum_{i=1}^{h} \frac{q_i}{p_i}.$$

The Euler number is only defined modulo $\mathbb{Z}$ when N has boundary. The good definition follows from Proposition 10.3.11 (the moves do not affect e, except (10) that modifies e into $e+1$ and applies only when N has boundary) and is coherent with the circle bundle case. The Euler number depends on the fibration and not only on N, but we write $e(N)$ anyway for simplicity. Proposition 10.3.11 easily implies the following.

Corollary 10.3.13. *Two Seifert fibrations*

$$(S, (p_1, q_1), \ldots, (p_h, q_h)), \quad (S', (p'_1, q'_1), \ldots, (p'_{h'}, q'_{h'}))$$

with $p_i, p'_i \geqslant 2$ *are orientation-preservingly isomorphic if and only if* $S = S'$, $h = h'$, $e = e'$, *and up to reordering* $p_i = p'_i$ *and* $q_i \equiv q'_i$ *(mod* p_i*) for all* i.

The numbers e and e' indicate the Euler numbers of the two fibrations, and recall that they are only defined modulo $\mathbb{Z}$ when $\partial S \neq \varnothing$.

Remark 10.3.14. The move

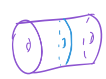
$$(S, (p_1, q_1), \ldots, (p_h, q_h)) \longmapsto (S, (p_1, -q_1), \ldots, (p_h, -q_h))$$

corresponds to a change of orientation for the three-manifold and transforms e into $-e$.

Proposition 10.3.11 classifies all the Seifert fibrations up to isomorphism. A classification of Seifert manifolds up to *diffeomorphism* would also be desirable, but it is much harder to obtain because a three-manifold may admit many non-isomorphic Seifert fibrations. For instance, Exercise 10.3.6 shows that the lens spaces may fibre in many different ways; a manifold as familiar as

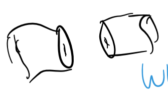
We easily understand when two Seifert fibrations are isomorphic. To classify Seifert manifolds up to diffeomorphism it only remains to understand which Seifert manifolds can have non-isomorphic fibrations. A long discussion has shown the following. We write $S^2 \times S^1$ as the lens space $L(0, 1)$.

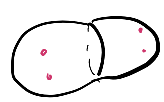
Theorem 10.4.19. *Every Seifert manifold has a unique Seifert fibration up to isomorphism, except the following:*

- $L(p, q)$ *fibres over* S^2 *with* $\leqslant 2$ *singular points in many ways,* lens spaces fall into equiv. classes
- $D \times S^1$ *fibres over* D *with* $\leqslant 1$ *singular point in many ways,* solid tori fiber in many ways
- $(D, (2, 1), (2, -1)) \cong S \tilde{\times} S^1$, $\longrightarrow$ read in book
- $(S^2, (2, 1), (2, -1), (p, q)) \cong (\mathbb{RP}^2, (q, p))$, relies on
- $(S^2, (2, 1), (2, 1), (2, -1), (2, -1)) \cong K \tilde{\times} S^1$. $\longrightarrow$ $K \tilde{\times} I$  $/\tau$ gives 2 cone points of order 2 in base

Here S *and* K *are the Möbius strip and the Klein bottle.*

Q: $K \tilde{\times} I \xrightarrow{\text{double}} K \tilde{\times} S^1$





Where are we going w/ all of this?

We want to show that the 6 classes in 10.3.26 actually define 6 of our geometries!