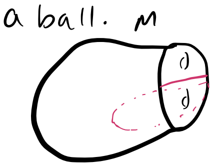
Recall Def: • M is <u>irreducible</u> if every sphere in M bounds a ball.



• Equivalently, M is irreducible if it doesn't contain essential spheres.

• M is <u>∂-irreducible</u> if it doesn't contain essential discs.

• A properly embedded orientable surface $S \subset M$ is <u>incompressible</u> if there are no compressing discs for S in M.

At end of last class we showed that for $T^2 \subset M$, T^2 is
(1) incompressible
(2) bounds a solid torus, or
(3) is contained in a ball.

Quick Skim:

9.3.3. ∂-incompressible surfaces. There is of course also a ∂-version of incompressibility. Let $S \subset M$ be a properly embedded orientable surface in a 3-manifold M. A *∂-compressing disc* for S is a disc D with $\partial D = \alpha \cup \beta$, where α lies in S and β in ∂M as in Figure 9.9-(left); we also require that there is no sub-disc $D' \subset S$ with $\partial D' = \alpha \cup \beta'$ and $\beta' \subset \partial S$. The move in Figure 9.9 is a *∂-compression* and transforms S into a surface $S' \subset M$ simpler than S:

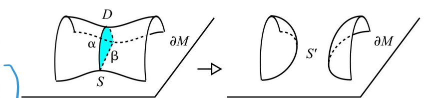
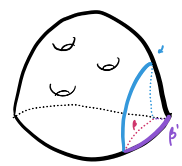


Figure 9.9. We can also surger a surface S along a disc D touching the boundary in a segment. The result is a new properly embedded surface S'.

we don't want α to be trivial in S.

avoiding:



b/c then compressing disc would give:

same surface w/ an extra disc.

Proposition 9.3.7. *The surface S' may have one or two components S'_i, and $\chi(S'_i) > \chi(S)$ for each component.*

Proof. We have $\chi(S') = \chi(S) + 1$. If S' has one component we are done, so suppose $S' = S'_1 \sqcup S'_2$. Since α did not bound a disc in S, no S'_i is a disc, hence $\chi(S'_i) \leqslant 0$ that implies $\chi(S'_i) > \chi(S)$ for $i = 1, 2$. $\square$

A properly embedded connected orientable compact $S \subset M$ with $\chi(S) \leqslant 0$ is *∂-compressible* if it has a ∂-compressing disc, and *∂-incompressible* otherwise. See Figure 9.16.

Corollary 9.3.8. *Let $S \subset M$ be any properly embedded orientable surface. After ∂-compressing it a finite number of times it transforms into a disjoint union of spheres, discs, and ∂-incompressible surfaces.*

Skip to § 9.4 Haken Manifolds:

Def: A <u>Haken manifold</u> is a compact, conn., oriented 3-mfd M w/ (poss. $\emptyset$) boundary which is irreducible, ∂-irreducible, and contains an incompressible and ∂-incompressible surface.

Proposition 9.4.2: Every boundary component X of a Haken manifold M has $\chi(X) \leq 0$ and is incompressible.

Pf: If a component X of ∂M is a sphere, then X bounds a ball $\Rightarrow M = B$ but there are **no** incompressible surfaces in a ball $\Rightarrow\Leftarrow$. Thus, $\chi(X) \leq 0$. But, M is ∂-irreducible so that M has **NO** essential discs, i.e. all discs are ∂-parallel. Thus, X is incompressible. ∎

Q: Are there even Haken mfds? Yes! lots of them

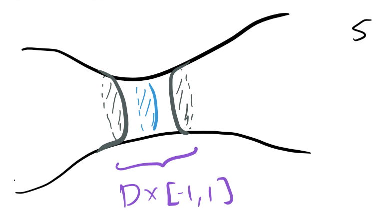
Proposition 9.4.3. *Let M be an oriented, compact, irreducible, and ∂-irreducible 3-manifold with (possibly empty) boundary. Every non-trivial homology class $\alpha \in H_2(M, \partial M; \mathbb{Z})$ is represented by a disjoint union of incompressible and ∂-incompressible oriented surfaces.*

Proof. Every class α is represented by a properly embedded oriented surface S by Proposition 1.7.16. A compression as in Figure 9.8 and 9.9 does not alter the homology class of the surface: indeed in homology we have $S' - S = \partial B$ where $B = D \times [-1, 1]$ is a tubular neighbourhood of the compressing disc D. Hence $[S'] = [S] = \alpha$.

We compress S until its connected components are either incompressible and ∂-incompressible surfaces, discs, or spheres. Since M is irreducible and ∂-irreducible, discs and spheres bound balls and are hence homologically trivial, so they can be removed. □

S' and S cobound a ball

Proposition 1.7.16. *Let M be a compact oriented n-manifold with (possibly empty) boundary. Every class in $H_{n-1}(M, \partial M; \mathbb{Z})$ is represented by an oriented properly embedded hypersurface $S \subset M$.*



Corollary 9.4.4. *Let M be oriented, compact, irreducible, and ∂-irreducible. If $H_2(M, \partial M; \mathbb{Z}) \neq \{e\}$ then M is Haken.*

Corollary 9.4.5. *Let M be oriented, compact, irreducible, and ∂-irreducible. If $\partial M \neq \varnothing$ and $M \neq B$, then M is Haken.*

Proof. If ∂M contains a sphere, it bounds a ball B and hence $M = B$. Otherwise $H_1(\partial M)$ has positive rank, and hence $H_2(M, \partial M) = H^1(M)$ also has positive rank by Corollary 9.1.5. □

Corollary 9.1.5. *Let M be an oriented compact 3-manifold. We have*

$$b_1(M) \geqslant \frac{b_1(\partial M)}{2}.$$

As we have already seen, every compact orient. 3-mfd decomposes along spheres and discs into irred. and ∂-irred. pieces. If one such piece has non-empty boundary, then it is either a ball, or it is Haken !!

So there are lots of Haken mfds.

But what about closed 3-mfds that are Haken? That's harder...

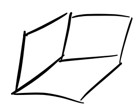
Heuristic: most "common" type of 3-mfd is hyperbolic.

<u>Virtual Haken Thm</u> (Agol '12): Let M be a closed, irreducible, 3-mfd with infinite fundamental group. Then, there is a finite-sheeted cover $\tilde{M} \to M$ s.t. $\tilde{M}$ is Haken.

includes hyperbolic mfds.

<u>tools</u>: ① work of Wise + collabs on $\pi_1(M) \curvearrowright$ CAT(0) c.c.

② Surface subgroup conj. by Kahn–Markovic



③ Malnormal special quotient thm of Wise

④ Cubulation criterion by Bergeron–Wise

⑤ hard work :")

Essentially Agol proves:

think $\pi_1(M)$ — hyp. 3-mfd.

$\pi_1(\tilde{M}) <_{f.i.} \pi_1(M)$

<u>Thm</u>: Let G be a word hyperbolic group acting properly and co-compactly on a CAT(0) cube complex. Then G has a finite index subgroup F acting specially on X.

$F_3 * \mathbb{Z}^2$

group theoretically $G' <_{f.i.} B$ embeds in a RAAG.

<u>Corollary</u>: Let G be a non-elem. word hyperbolic group acting properly and cocompactly on a CAT(0) c.c. Then G is

linear, large, and quasi-convex subgroups are separable.

injects into $GL_n(K)$

has a finite index subgroup that surjects F_2

next time

In fact if $\pi_1(\tilde{M}) \underset{f.i.}{<} \pi_1(M)$ embeds in RAAG, $\tilde{M}$ is Haken because

$\pi_1(M)$ large $\Rightarrow b_1(\tilde{M}) = b_2 > 0$.

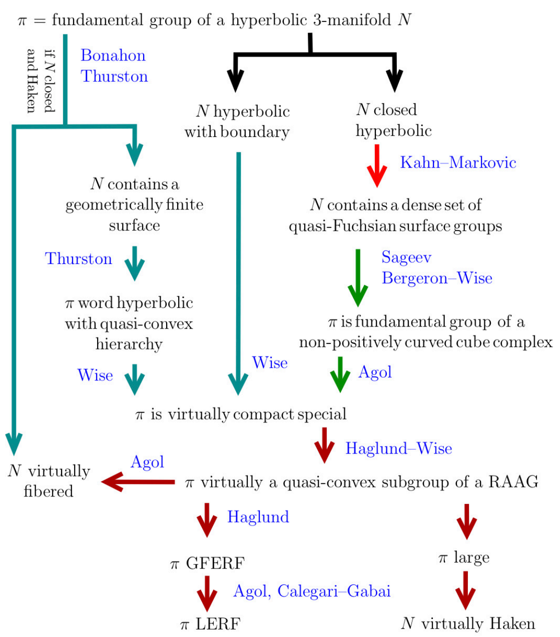


Diagram 2. The Virtually Compact Special Theorem.

Open:

Q: Given M hyp. closed mfd, what is the degree of the smallest Haken cover?

aka

Given $\pi_1(M)$ what is the smallest index of a finite index special subgroup?