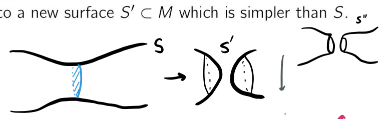
9.3. Incompressible surfaces

We have proved that every compact oriented 3-manifold decomposes along essential spheres and discs, into some canonical pieces that do not contain essential spheres or discs anymore.

We would like to pursue this strategy with the next simplest surfaces, namely tori and annuli. To this purpose we define the important notion of *incompressible* surface in a three-manifold, which applies to all surfaces of non-positive Euler characteristic.

9.3.1. Incompressible surfaces. Throughout all this section M denotes a compact orientable 3-manifold with (possibly empty) boundary. Let $S \subset M$ be a properly embedded orientable surface. A *compressing disc* for S is a disc $D \subset M$ with $\partial D = D \cap S$, such that ∂D does not bound a disc in S. With this hypothesis, the surgery in Figure 9.8 is called a *compression*: it transform S into a new surface $S' \subset M$ which is simpler than S.



$\chi(S) = \chi(S'')$
$\chi(S') = \chi(S) + 1 + 1$
$= \chi(S) + 2$

$\chi(D) = 1 - 1 + 1 = 1$
$\chi(S') = 0$

Proposition 9.3.1. *The surface S' may have one or two components S'_i, and $\chi(S'_i) > \chi(S)$ for each component.*

Proof. We have $\chi(S') = \chi(S) + 2$. If S' has one component we are done, so suppose $S' = S'_1 \sqcup S'_2$. Since ∂D did not bound a disc in S, no S'_i is a sphere, hence $\chi(S'_i) \leqslant 1$ that implies $\chi(S'_i) > \chi(S)$ for $i = 1, 2$. □

Suppose $\chi(S'_1) \leq \chi(S)$ or $\chi(S'_2) \leq \chi(S)$ $\qquad$ $\chi(S) + 2 = \chi(S'_1) + \chi(S'_2) \leq \chi(S)$

$\chi(S) + 2 = \chi(S'_1) + \chi(S'_2) \leq \chi(S) + 1 \Rightarrow\!\Leftarrow$

A properly embedded connected orientable compact surface $S \subset M$ with $\chi(S) \leqslant 0$ is *compressible* if it has a compressing disc, and *incompressible* otherwise. See Figure 9.16-(top).

Corollary 9.3.2. *Let $S \subset M$ be any properly embedded orientable surface. After compressing it a finite number of times it transforms into a disjoint union of spheres, discs, and incompressible surfaces.*

Proof. We compress S as much as we can; after finitely many steps we must stop because of Proposition 9.3.1. □

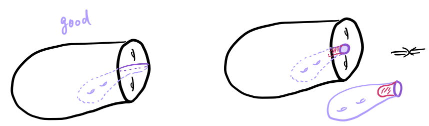
A simple incompressibility criterion is the following.

Proposition 9.3.4. *Let $S \subset M$ be an orientable, connected, properly embedded surface with $\chi(S) \leqslant 0$. If the map $\pi_1(S) \to \pi_1(M)$ induced by inclusion is injective, then S is incompressible.*

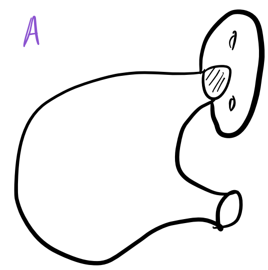
The converse is also true, but its proof is much harder! We will complete it at the end of this chapter. For the moment we content ourselves with the following.

Proposition 9.3.5. *If $S \subset M$ is incompressible, every component of ∂S is non-trivial in ∂M.*

Proof. If a component of ∂S is trivial in ∂M, it bounds a disc $D \subset \partial M$ there. By taking an innermost one we get $D \cap S = \partial D$, and by pushing D inside M we find a compressing disc for S. □



Let's consider incompressible surfaces in 3-mfds starting with the simplest: T^2 and A



9.3.2. Tori. The first closed surface to look at is the torus.

Proposition 9.3.6. *Let $T \subset M$ be a torus in an irreducible 3-manifold. One of the following holds:*

(1) *T is incompressible,*
(2) *T bounds a solid torus,*
(3) *T is contained in a ball.*

Proof. If T is not incompressible, it compresses along a disc D. The result of the compression is necessarily a sphere $S \subset M$ which bounds a ball B since M is irreducible. If B is disjoint from T, then T bounds a solid torus as in Fig.9.17-(left). If B contains T, then case (3) holds as shown in Figure 9.17-(right). □