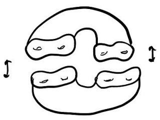
10.2. Circle bundles

We now introduce another simple class of 3-manifolds, the orientable circle bundles over some compact surface S. We will discover that there is essentially only one circle bundle if S has boundary, and infinitely many if S is closed, distinguished by an integer called the *Euler number*.

10.2.1. The trivial circle bundle. Let S be a compact connected surface. As every connected manifold, the surface S has a unique orientable line bundle

$$S \times I \quad \text{or} \quad S \mathbin{\tilde{\times}} I$$



depending on whether S is orientable or not. We denote by

$$M = S \times S^1 \quad \text{or} \quad S \mathbin{\tilde{\times}} S^1$$

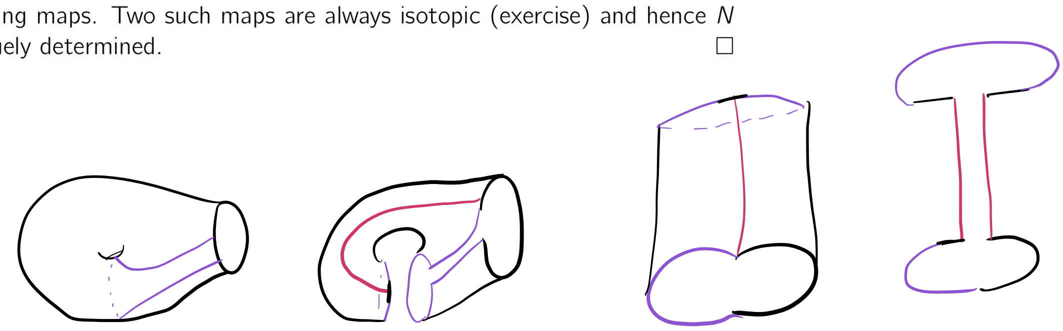
respectively the double of $S \times I$ and $S \mathbin{\tilde{\times}} I$ along its boundary. If we do not know whether S is orientable or not, we use the symbols $S \mathbin{(\tilde{\times})} I$ and $S \mathbin{(\tilde{\times})} S^1$ to denote these objects. The manifold $S \mathbin{(\tilde{\times})} S^1$ is an orientable circle bundle over S, called the *trivial* one.

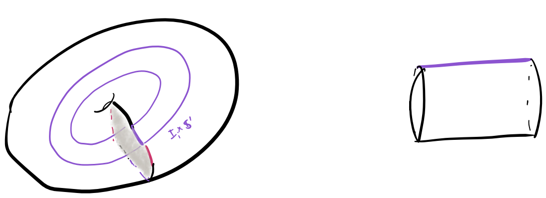
10.2.2. Circle bundles with boundary. We start by exploring the case where the base surface S has non-empty boundary: in this case every bundle M over S is a 3-manifold with boundary; the boundary consists of tori, one fibering above each circle in ∂S, because the torus is the unique orientable surface that fibres over S^1.

It turns out that there is essentially only one bundle over S, the trivial one:

Lemma 10.2.1. *If $\partial S \neq \varnothing$, the orientable circle bundles on S are all isomorphic.*

Proof. Let $N \to S$ be an orientable circle bundle. Decompose S as a disc D with some pairs of disjoint segments (I_i, J_i) in ∂D to be glued. Since D is contractible the restriction of N to D is a product $D \times S^1$ and N is obtained from it by gluing the annuli $I_i \times S^1$ and $J_i \times S^1$ via orientation-reversing fibre-preserving maps. Two such maps are always isotopic (exercise) and hence N is uniquely determined. $\square$

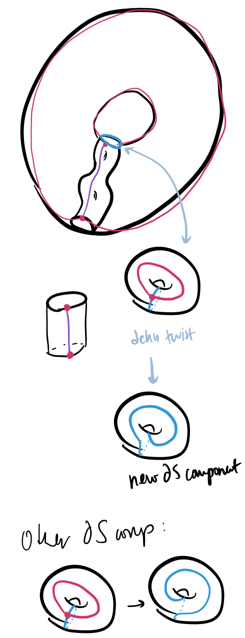




We now want to study the *sections* of the trivial bundle $M \to S$, because these will be useful in the study of bundles over closed surfaces. We now discover that, although the bundle is trivial, it contains many non isotopic sections, and we want to classify them.

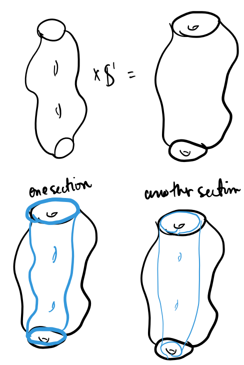
Recall that a *section* of the bundle $\pi\colon M \to S$ is a map $i\colon S \to M$ such that $\pi \circ i = \text{id}$. Since the section i is determined by its image $i(S)$, we simply consider the surface $i(S)$ as a section of π. By construction M is the double of an interval bundle over S and as such it contains the *zero-section* S there. However, this section is not unique in general, not even up to isotopy.

To modify a section, pick a properly embedded arc in S. The arc determines a fibered annulus $A \subset M$ above it, which we may use to *twist* the section as shown in Figure 10.8. This operation modifies the curves $\partial S \subset \partial M$ via two Dehn twists (one positive and one negative) on the tori ∂M along the two curves in ∂A.



By twisting along annuli we may construct all the sections of M:

Lemma 10.2.2. *Two sections of $S \overset{(\sim)}{\times} S^1$ are connected by a composition of twists along fibered annuli and fibre-preserving isotopies.*



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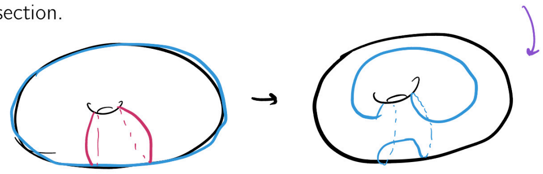
Corollary 10.2.3. *If S has only one boundary component, the boundary of a section of $M = S \overset{(\sim)}{\times} S^1$ is a slope in ∂M that does not depend on the section.*

Proof. Distinct sections are connected by finitely many Dehn twists along annuli. One such twist acts on the torus ∂M as a composition of two opposite Dehn twists, which cancel each other. Hence it does not affect the boundary slope of a section. $\square$



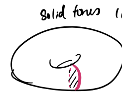
10.2.3. Closed circle bundles. We now turn to closed circle bundles. In this section we prove that the oriented circle bundles over a closed surface are parameterised by an integer called the *Euler number* of the fibering.

We prefer to see the bundles over closed surfaces as Dehn fillings of bundles over surfaces with boundary. Here are the details.

Let S be a compact surface with non-empty boundary. Pick $M = S \overset{(\sim)}{\times} S^1$ and fix an orientation for M. Recall that we denote by S the zero-section of M. Every boundary component T of M is an oriented torus, which contains two natural unoriented simple closed curves: the boundary $m = T \cap \partial S$ of the section S and the fibre l of the bundle. If oriented, the curves m and l form a basis (m, l) for $H_1(T, \mathbb{Z})$. We choose orientations for m and l such that (m, l) form a positively oriented basis: there is a unique choice up to reversing both m and l.

here $l = \binom{1}{0}$ $m = \binom{0}{1}$

Solid torus label $m' = \partial D$ and $l' = \{x\} \times S^1$

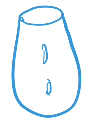


Let $m' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $l = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

• $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} q \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$: $1\,m' \mapsto 1 \cdot m + 2 \cdot l$

$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$: $1\,m' \mapsto 1 \cdot m + 3 \cdot l$

$\left.\begin{array}{l} A^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} : \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} : \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}\right\}$ these are choice of fiber in $D^2 \times S^1$

Note that quotienting fiber in $D^2 \times S^1$ simply yields D^2.

A Dehn filling on T is determined as usual by a pair (p, q) of coprime integers that indicate the slope $\pm(pm + ql)$ to be killed.

Suppose now that S has only one boundary component and let M^{fill} be obtained by Dehn filling M along the slope $(1, q)$. Let $\hat{S}$ be the closed surface obtained by capping S with a disc.

On the other hand:

$m' = 3 \cdot m + 2 \cdot l$

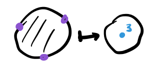
$C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} : \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$C^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} : \overset{l}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \mapsto \overset{2m' - 3l'}{\begin{bmatrix} 2 \\ -3 \end{bmatrix}}$

Proposition 10.2.5. *The circle bundle $M \to S$ extends to a circle bundle $M^{\text{fill}} \to \hat{S}$. Every oriented circle bundle on $\hat{S}$ is obtained in this way, and distinct values of q yield vector bundles that are not orientation-preservingly isomorphic.*

Proof. The meridian of the filling solid torus is $m' = m + ql$. The fibre l has geometric intersection 1 with m' and is hence a longitude for the filling solid torus. We may represent the filling solid torus as $D \times S^1$ with $m' = S^1 \times \{y\}$ and $l = \{x\} \times S^1$. The circle bundle $M \to S$ extends naturally to a circle bundle $M^{\text{fill}} \to \hat{S}$ with $\hat{S} = S \cup D$.

Every closed circle bundle $N \to \hat{S}$ arises in this way: the bundle above a disc $D \subset \hat{S}$ is the trivial $D \times S^1$, and if we remove it we get $M \to S$ back. The number q is intrinsically determined: the meridian m does not depend on the section of $M \to S$ by Corollary 10.2.3, and the equality $m' = m + ql$ determines q. Therefore distinct values of q yield non-isomorphic bundles. $\square$

Collapse circle fiber $\Rightarrow$ 

S^1 bundle over S in unique and diff. choices of q can be seen from "within" M^{fill} so distinguish these mfds.