Chapter 10 Seifert Manifolds

These are generalizations of circle bundles over surfaces where we allow for "singular" fibers.

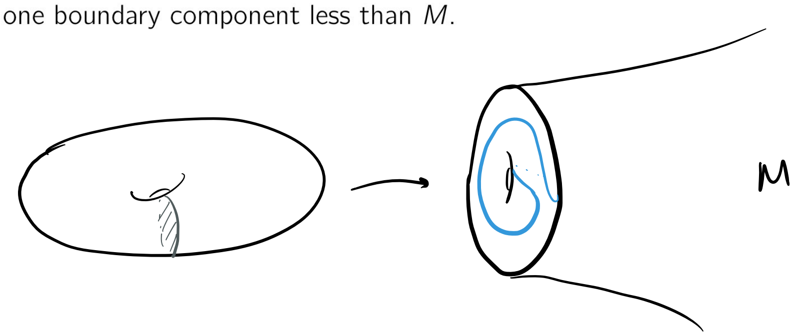
We start with Lens spaces and for that we need Dehn filling

10.1.1. Dehn filling. If a 3-manifold M has a spherical boundary component, we can cap it off with a ball. If M has a toric boundary component, there is no canonical way to cap it off: the simplest object that we can attach to it is a solid torus $D \times S^1$, but the resulting manifold depends on the gluing map. This operation is called a *Dehn filling* and we now study it in detail.

- Let M be a 3-manifold and $T \subset \partial M$ be a boundary torus component.

Definition 10.1.1. A *Dehn filling* of M along T is the operation of gluing a solid torus $D \times S^1$ to M via a diffeomorphism $\varphi\colon \partial D \times S^1 \to T$.

The closed curve $\partial D \times \{x\}$ is glued to some simple closed curve $\gamma \subset T$, see Figure 10.1. The result of this operation is a new manifold M^{fill}, which has one boundary component less than M.



Lemma 10.1.2. *The manifold M^{fill} depends only on the isotopy class of the unoriented curve γ.*

Proof. Decompose S^1 into two closed segments $S^1 = I \cup J$ with coinciding endpoints. The attaching of $D \times S^1$ may be seen as the attaching of a 2-handle $D \times I$ along $\partial D \times I$, followed by the attaching of a 3-handle $D \times J$ along its full boundary.

If we change γ by an isotopy, the attaching map of the 2-handle changes by an isotopy and hence gives the same manifold. The attaching map of the 3-handle is irrelevant by Proposition 9.2.1. □

Proposition 9.2.1. *The manifold M obtained by capping off a spherical boundary component of N does not depend on the diffeomorphism chosen.*

We say that the Dehn filling *kills* the curve γ, since this is what really happens on fundamental groups, as we now see.

The *normaliser* of an element $g \in G$ in a group G is the smallest normal subgroup $N(g) \triangleleft G$ containing g. The normaliser depends only on the conjugacy class of $g^{\pm 1}$, hence the subgroup $N(\gamma) \triangleleft \pi_1(M)$ makes sense without fixing a basepoint or an orientation for γ.

Proposition 10.1.3. *We have*

$$\pi_1(M^{\text{fill}}) = \pi_1(M)/_{N(\gamma)}.$$

Proof. The Dehn filling decomposes into the attachment of a 2-handle over γ and of a 3-handle. By Van Kampen, the first operation kills $N(\gamma)$, and the second leaves the fundamental group unaffected. $\square$

Let a *slope* on a torus T be the isotopy class γ of an unoriented homotopically non-trivial simple closed curve. The set of slopes on T was indicated by $\mathscr{S}$ in Chapter 7. If we fix a basis (m, l) for $H_1(T, \mathbb{Z}) = \pi_1(T)$, every slope may be written as $\gamma = \pm(pm + ql)$ for some coprime pair (p, q). Therefore we get a 1-1 correspondence

$$\mathscr{S} \longleftrightarrow \mathbb{Q} \cup \{\infty\}$$

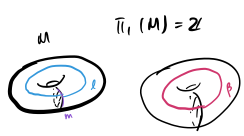
by sending γ to $\frac{p}{q}$. If T is a boundary component of M, every number $\frac{p}{q}$ determines a Dehn filling of M that kills the corresponding slope γ.

Different values of $\frac{p}{q}$ typically produce non-diffeomorphic manifolds M^{fill}: this is not always true - a notable exception is described in the next section - but it holds in "generic" cases.

10.1.2. Lens spaces. The simplest manifold that can be Dehn-filled is the solid torus $M = D \times S^1$ itself. The oriented *meridian* $m = S^1 \times \{y\}$ and *longitude* $l = \{x\} \times S^1$ form a basis for $H_1(\partial M, \mathbb{Z})$.

Definition 10.1.4. The *lens space* $L(p, q)$ is the result of a Dehn filling of $M = D \times S^1$ that kills the slope $qm + pl$. note the swap

A lens space is a three-manifold that decomposes into two solid tori. We have already encountered lens spaces in the more geometric setting of Section 3.4.10, and we will soon prove that the two definitions are coherent. Since $L(p, q) = L(-p, -q)$ we usually suppose $p \geqslant 0$.

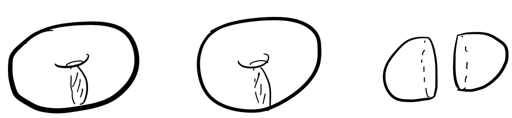


$|z_1|^2 + |z_2|^2 = 1$

Exercise 10.1.5. We have $\pi_1(L(p, q)) = \mathbb{Z}/_{p\mathbb{Z}}$. $\longrightarrow$ for students during class

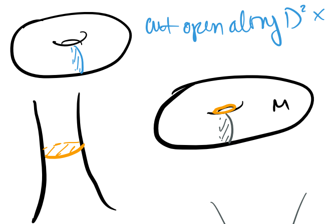
Proposition 10.1.6. *We have* $L(0, 1) = S^2 \times S^1$ *and* $L(1, 0) = S^3$.

Proof. The lens space $L(0, 1)$ is obtained by killing m, that is by mirroring $D \times S^1$ along its boundary. The lens space $L(1, 0)$ is S^3 because the complement of a standard solid torus in S^3 is another solid torus, with the roles of m and l exchanged (exercise). $\square$ **

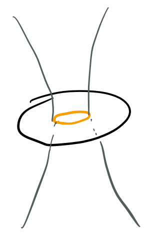


$L(0,1)$ meridian glues to meridian

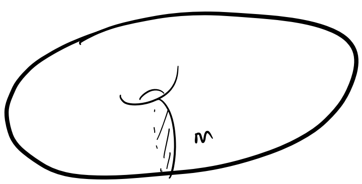
Another way of constructing a lens space is Dehn filling each boundary of $T^2 \times I$

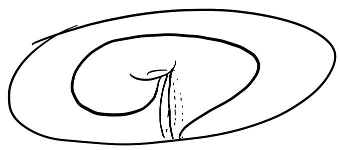


Remark 10.1.9. The meridian m of the solid torus $M = D \times S^1$ may be defined intrinsically as the unique slope in ∂M that is homotopically trivial in M. The longitude l is *not* intrinsically determined: a twist sends l to $m + l$. The solid torus contains infinitely many non-isotopic longitudes, and there is no intrinsic way to choose one of them.



keep id-parallel orange curves and we will get $\mathbb{R}^3$. Next, blue disc we cut out is at ∞ so $L(1,0) = S^3 = \mathbb{R}^3 \cup \{\infty\}$





a possible longitude $3m+l$

Theorem 10.1.12. *The lens spaces* $L(p,q)$ *and* $L(p',q')$ *are diffeomorphic* $\iff p = p'$ *and* $q' \equiv \pm q^{\pm 1}$ *(mod* p*).*

$L(3,1)$ and $L(3,4)$ are diffeo!

$L(5,1)$ and $L(5,2)$ are not even though they have same homotopy and homology groups.