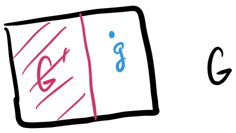
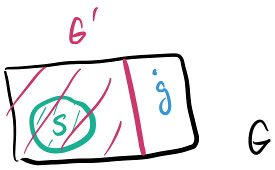
Def: G is residually finite if $\forall\ g \in G - \{id\}$, $\exists\ G' <_{f.i.} G$ s.t. $g \notin G'$.



G' separates g from id.

Alt. def: G is RF if $\forall\ g \in G - \{id\}$, $\exists\ \varphi : G \to F$ where F is finite and $g \notin \ker(\varphi)$

Def: A subgroup S of G is separable if $\forall\ g \in G - S$ $\exists\ G' <_{f.i.} G$ s.t. $S < G'$ but $g \notin G'$.



G' separates S from g.

If G is separable on all finitely gen subgroups then G is LERF (locally extended res. fin)

QCERF (sep on quasi-convex subgroups)

GFERF (geom. finite subgroups are separable)

RF and algorithms: Word problem for a group — can you decide (in finite time) whether an element $g \in G$ is the trivial elmnt.

RF $\Rightarrow$ solvable word problem

$GL_2(\mathbb{Z})$ is RF : take $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq id$

choosing p carefully we see that $A \notin \ker(\varphi_p)$ where
$\varphi_p : GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}_p)$

ex : $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ then we can choose $p = 5$ for instance.

Exercise : Prove that if G is RF then any subgroup of G is RF and any finite index extension of G is also RF.
(i.e. $H \underset{f.i.}{>} G$)

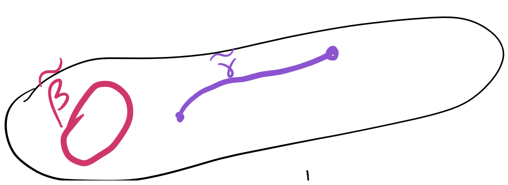
Let $H < G$ and take $h \in H - id$
$\Rightarrow \exists\ G' \underset{f.i.}{<} G$ and $h \notin G'$. Now use the following fact:

$[G' \cap H : H] \leq [G' : G] \Rightarrow G' \cap H$ is the f.i. subgroup of H s.t. $h \notin G' \cap H$.

What does RF mean geometrically:

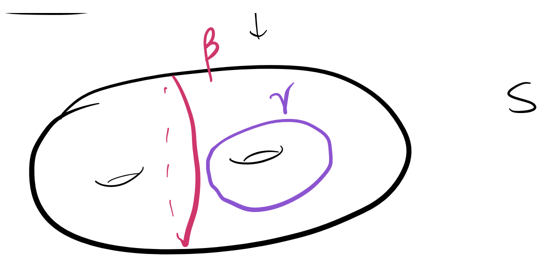
Consider $\pi_1(S)$ S is a surface
$\gamma \in \pi_1(S)$ where $\gamma \neq id \Rightarrow \exists\ \pi_1(\tilde{S}) \underset{f.i.}{<} \pi_1(S)$
s.t. $\gamma \notin \pi_1(\tilde{S})$. Here $\tilde{S}$ is finite deg. cover of S.

Fact :
① $\pi_1(S)$ is RF



$\tilde{S}$

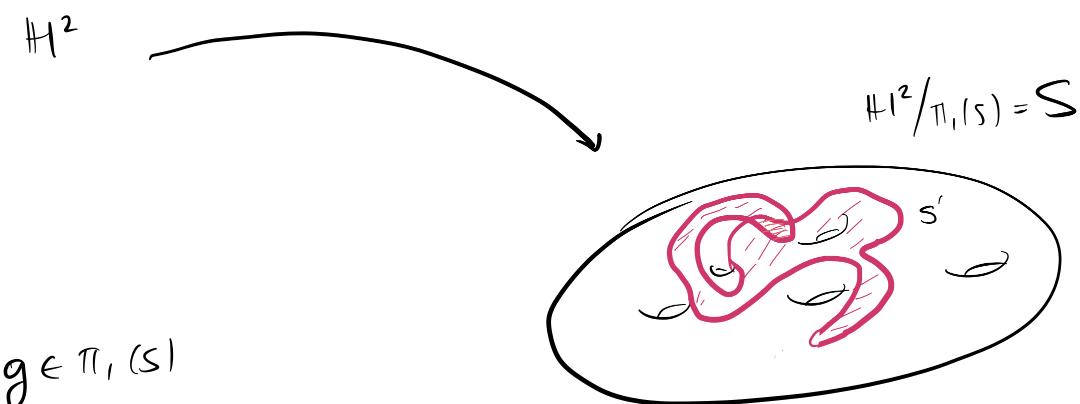
$\beta \in \pi_1(\tilde{S})$



Scott '86 ish: $\pi_1(S)$ are LERF

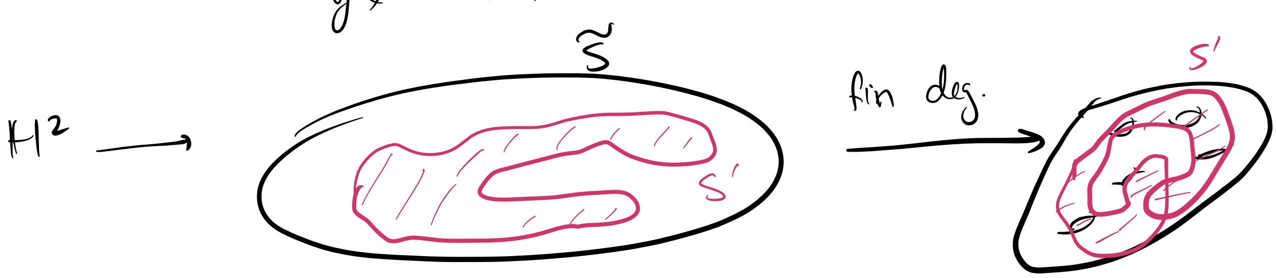
Scott's Criterion

$\pi_1(S)$ being LERF means that we can promote immersions to embeddings using finite covers.



$\exists\, g \in \pi_1(S)$ that causes C to bump into itself

If I find $\pi_1(\tilde{S}) \le_{f.i.} \pi_1(S)$ s.t. $\pi_1(S') < \pi_1(\tilde{S})$ but $g \notin \pi_1(\tilde{S})$



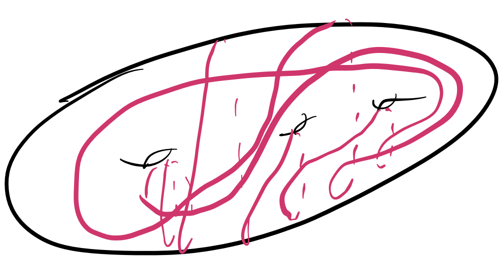
In general, $g_1, \dots, g_k \in \pi_1(S)$ causing a cpt set S' to intersect itself: so applying LERF-ness k times gives us a $\pi_1(\tilde{S}) <_{f.i.} \pi_1(S)$

s.t. $\pi_1(S') < \pi_1(\tilde{S})$ but $g_1, \dots, g_k \notin \pi_1(\hat{S})$

Scott: Let γ be a closed geodesic in S, then $\exists\ \tilde{S} \to S$ fin. deg. s.t. γ lifts to be embedding (i.e. $\gamma \in \pi_1(\tilde{S})$ and γ is a s.c.c.)

If we don't care about a fin. deg. cover there is a more obvious covering space of S s.t. γ is a s.c.c. in this cover.

Consider the cover of S corr. to $\langle \gamma \rangle < \pi_1(S)$.



$\gamma \in \langle \gamma \rangle$

γ is coxod.

Let's consider LERF prop: $\exists\, g_1, \dots, g_k$ causing self int. in γ $\quad g_1, \dots, g_k \notin \langle \gamma \rangle$

using LERF we sep. $g_1, \dots g_k$ from $\langle \gamma \rangle$ using f.i. subgroup $\pi_1(\hat{S}) \underset{\text{f.i.}}{<} \pi_1(S)$.

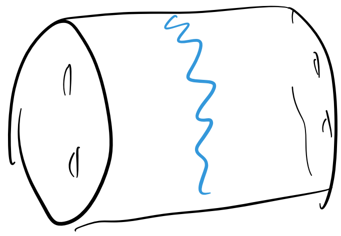
$\hat{S}$ is the cover we are looking for.

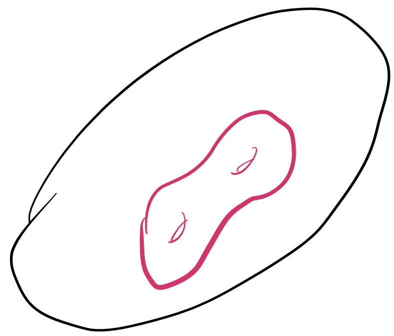
Now consider $\pi_1(M)$ where M closed hyp 3-mfd. <u>If</u> we knew that $\pi_1(M)$ was LERF (or QCERF) then we win:

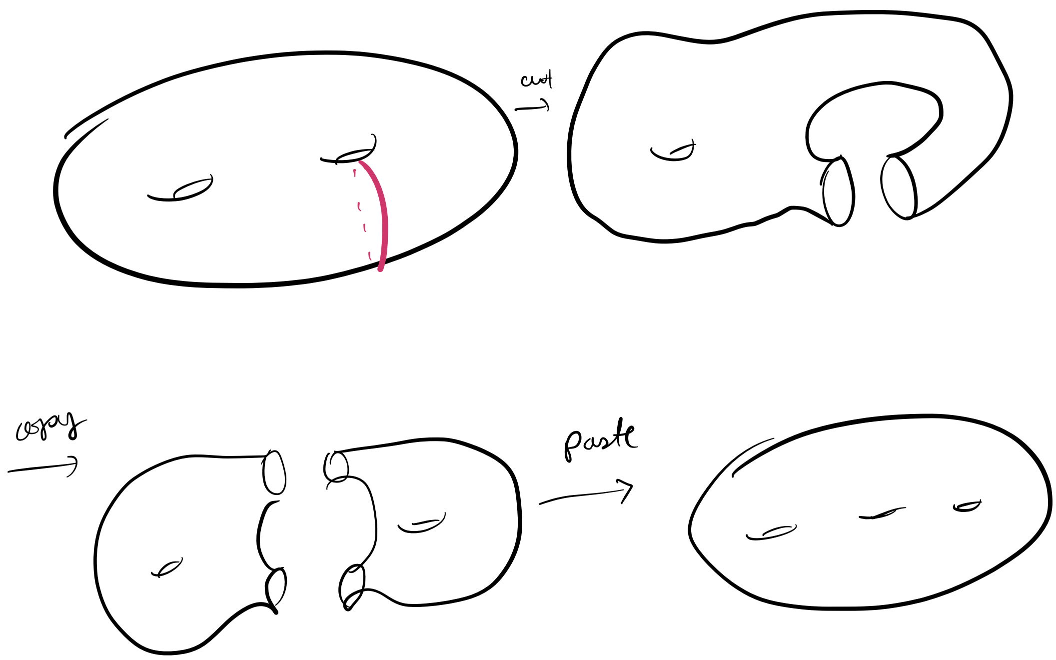
① Kahn-Markovic $\Rightarrow$ there are <u>lots</u> of quasi-convex immersed surfaces in any closed hyp. 3-mfd.

② Scott's criterion tells us that QCERF allows us to promote such an immersed surface to be embedded in a fin. deg cover.

RAAG are RF and QCERF



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