Recall that for $\mathbb{E}^3$ and Nil we have $\chi(B)=0$ and the possible orbifolds B are: T, K, $(\mathbb{RP}^2, 2, 2)$, $(S^2, 2, 2, 2, 2)$, $(S^2, 2, 3, 6)$, $(S^2, 3, 3, 3)$, $(S^2, 2, 4, 4)$.

- From this we saw that for $\mathbb{E}^3$ mfds, the condition that $e=0$ gives us only 6 $\mathbb{E}^3$ mfds.
- For Nil we can do many diff fillings to get $e \neq 0$ for instance when $B = (S^2, 2, 3, 6)$, a Nil mfd M could be $M = (S^2, (2,1), (3,1), (6,1))$

or

$(S^2, (2,1), (3,2), (6,1))$

or

$(S^2, (2,1), (3,2), (6,5))$

or

$(S^2, (2,3), (3,2), (6,1))$

etc. ...

In this way we see that if the list of poss. base orbifolds is finite, we can characterize mfds M without too much difficulty.

Then, for S^3 geometry, we had the next best case. The possible B subject to $\chi(B)>0$ were

(S^2, n, n), $(S^2, 2, 2, n)$, $(S^2, 2, 3, 3)$, $(S^2, 2, 3, 4)$, $(S^2, 2, 3, 5)$

<u>and</u> we needed to ensure that the slopes of fillings were s.t.

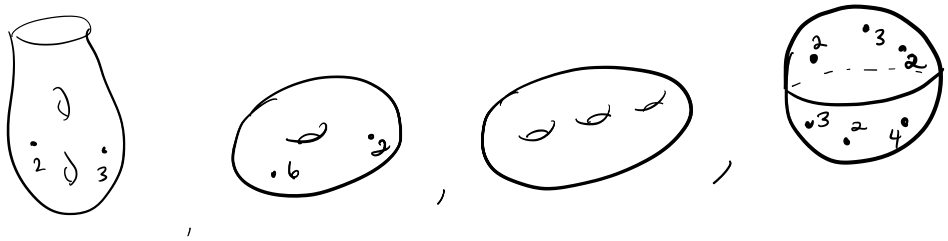
$e = \sum \frac{q_i}{p_i} \neq 0$. For (S^2, n, n) and $(S^2, 2, 2, n)$ families (lens spaces and prism mfds) we ensured this using reg. language $\langle L, R\rangle \cdot R$ where R ensured meridian wasn't glued to meridian.

Now we want to intuitively understand $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL_2}$ geoms assuming we know:

- M has $\mathbb{H}^2 \times \mathbb{R}$ geom. $\iff$ $e = 0$ and $\chi(B) < 0$ (last class)
- M has $\widetilde{SL_2}$ geom $\iff$ $e \neq 0$ and $\chi(B) < 0$.

First observation is that orbifolds w/ $\chi(B) < 0$ aren't as easily characterized as above.

exs:



SO MANY!

Once we get a handle on these, we then understand $\mathbb{H}^2 \times \mathbb{R}$ geom vs $\widetilde{SL_2}$ as which fillings give $e = \sum \frac{q_i}{p_i} = 0$ or $\neq 0$.

From perspective of project I'm working on, we want to construct branched mfds $M_{\mathbb{H}^2 \times \mathbb{R}}$ and $M_{\widetilde{SL_2}}$ s.t.

closed mfd M immerses in $M_{\mathbb{H}^2 \times \mathbb{R}} \iff M$ admits $\mathbb{H}^2 \times \mathbb{R}$ geom

"   " in M_{SL_2} $\Longleftarrow$ "   " $\widetilde{SL}_2$ geom.

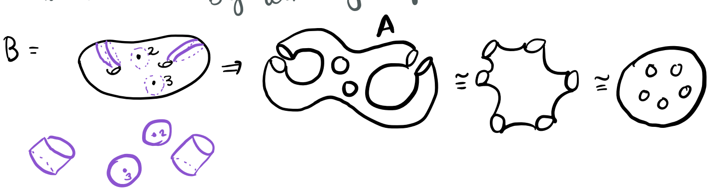
First let's try to understand all B w/ $\chi(B)<0$.

<u>Def</u>: There is a compact, connected, planar surface $A \subset B$ called the <u>scaffolding</u> that is disjoint from the singular set of B and s.t. each conn. comp W of $\overline{B \setminus A}$ is:

- disc w/ a cone pt., } create sing. locus
- a Möbius band (add #$\mathbb{RP}^2$'s), or
- an annulus (to create genus) } create non planar topology

We call these possible W's scaffolding attachments since we obtain B from A by attaching copies of the W's

<u>ex</u>: Let B =



W's :

Now let m = # of ∂ comps of A $\Rightarrow$ $\chi(A) = 2-m$. Let $p_1, \dots, p_k$ be orders of cone pts of B so that $k \leq m$. Then,

$$\chi(B) = \chi(|B|) - \sum_{i=1}^{k}\left(1-\frac{1}{p_i}\right) = 2-m+k-k+\sum_{i=1}^{k}\frac{1}{p_i} = 2-m+\sum_{i=1}^{k}\frac{1}{p_i}$$

Since $p_i \geq 2$, we have

$$2-m \leq \chi(B) \leq 2-\frac{m}{2}$$

ex: Show that other attachments don't effect $\chi(A)$ vs $\chi(B)$.

Back to goal: we want to understand which B have $\chi(B)<0$.

① $m \geq 3$ otherwise $2-m \leq \chi(B) \Rightarrow \chi(B) \geq 0$.

② If $m \geq 5$ then # of cone pts and their orders don't matter

(because $\chi(B) = 2-m+\sum_{i=1}^{k}\frac{1}{p_i} \leq 2-m+\sum_{i=1}^{k}\frac{1}{2}$ where $k \leq m$

(on other hand when $m=4$ and $\exists$ 4 cone pts of order 2, $\chi(B) = 2-4+2=0$, which we want to exclude)

So when $m = \#\,\partial$ comps of $A \geq 5$ we call B generic.

when $m = 3,4$, we call $\underbrace{B \text{ exceptional}}_{\text{sign of } \chi(B) \text{ depends on order of cone pts.}}$.

<u>Note</u>: Since $\chi(B) = 2 - m + \sum_{i=1}^{k} \frac{1}{p_i}$, if we increase the order of a cone pt p_i or decrease the total no. of cone pts k of B, this only reduces $\chi(B)$.

There are seven exceptional orbifolds B w/ $\chi(B)$ that we list now using the notation $(m, k; p_1, \dots, p_k)$

m: # ∂ comp of A

k: # of cone pts

p_i: order of cone pt. if $p_i \geq 2$ we leave it off the list

The seven exceptional orbifolds are:

$(4, \leq 3)$, $(4,4; \geq 3)$, $(3,3; \geq 7, \geq 3)$, $(3,3; \geq 5, \geq 4)$

$(3,3; \geq 4, \geq 3, \geq 3)$ $(3,2; \geq 3)$ $(3, \leq 1)$

$(4, \leq 3; \geq 2, \geq 2, \geq 2)$

$(4,4; \geq 3, \geq 2, \geq 2, \geq 2)$

$(3,3; \geq 4, \geq 3, \geq 3)$

$|\partial A|$ 3 cone pts   orders

Let's focus on generic case. So we consider all scaffoldings w/ ≥ 5 ∂ comps.

For project we need to find a way to encode this huge class of orbifolds concisely $\longrightarrow$ use branched surface!

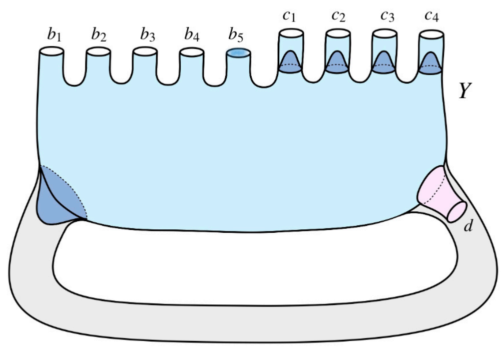


Figure 2: The branched scaffolding Y

Claim: All scaffoldings $A = A(m)$ w/ $m \geq 5$ immerse in Y.

In fact the only cpt. surfaces that immerse in Y are $A(m)$ w/ $m \geq 5$.

ex:

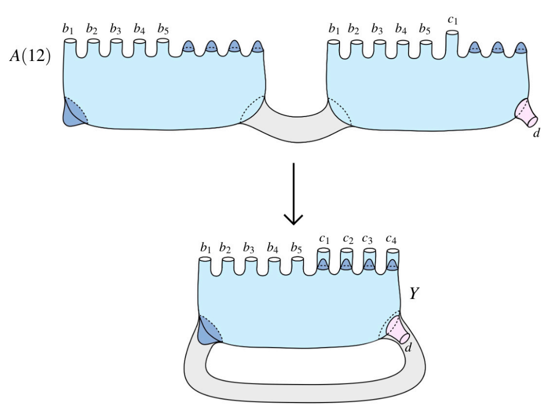


Figure 3: Immersion of $A(12)$ into Y

Aim: Take $Y \times S^1$