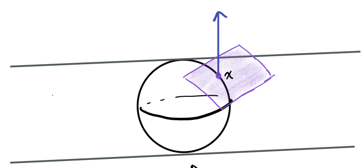
Product Geometries : $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

$S^2 \times \mathbb{R}$: (very few of these)

(our goal) : Claim : $\mathrm{Isom}(S^2 \times \mathbb{R}) = \mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R})$

Sec curvature is a no. assigned to every plane in tangent space of every pt $p \in S^2 \times \mathbb{R}$

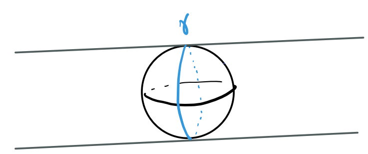
- plane is horizontal if its tangent to S^2 factor
- vertical if its tangent to $\mathbb{R}$ factor



(x,y)

$S^2 \times \mathbb{R}$

Prop 12.4.1 : The sectional curv. of horiz. and vert. planes are 1 and 0, resp.



$S^2 \times y$ reflect in $\mathbb{R}$ coord about y

Proof. Let $\gamma \subset S^2$ be a closed geodesic. The surfaces $S^2 \times y$ and $\gamma \times \mathbb{R}$ are totally geodesic, because they are fixed by some isometric reflections of $S^2 \times \mathbb{R}$. Therefore the sectional curvatures of horizontal and vertical planes equal the gaussian curvatures of these surfaces, which are 1 and 0. $\square$

Proposition 12.4.2. *We have*

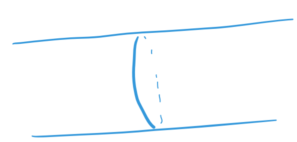
$$\mathrm{Isom}(S^2 \times \mathbb{R}) = \mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R}).$$

$S^2 \times \mathbb{R} / \Gamma$

$p = (x,y)$

Proof. We certainly have the inclusion $\supset$, which gives to every point $p \in S^2 \times \mathbb{R}$ a stabiliser in $\mathrm{Isom}^+(S^2 \times \mathbb{R})$ isomorphic to $SO(2) \times C_2$, a proper maximal subgroup of SO(3) by Proposition 6.2.15.

If there were more isometries that that, there would be more fixing p since they act transitively on $S^2 \times \mathbb{R}$ and the stabiliser would be the whole of SO(3), a contradiction because the sectional curvature of $S^2 \times \mathbb{R}$ is not constant. $\square$



$\gamma \times \mathbb{R}$

reflect in sphere coord. about γ

The isom. groups of S^2 and $\mathbb{R}$ each have two conn. components, $\therefore$ the group $\mathrm{Isom}(S^2 \times \mathbb{R})$ has 4 conn. comps, two of which are orient. pres.

Orient pres. isom group of a cylinder

$x \in S^2$ is stabilized by $SO(2) \subset O(3)$

C_2 acts like refle. on $\mathbb{R}$

Prop. 12.4.3: An orientable mfd. M admits a finite vol. $S^2 \times \mathbb{R}$ geom iff M is a closed Seifert mfd with $\chi > 0$ and $e = 0$.

Pf: The closed Seif. mfds w/ $e = 0$ and $\chi > 0$ are $S^2 \times S^1$ and $\mathbb{RP}^2 \tilde{\times} S^1$.

We want to classify the closed Seifert manifolds with $\chi \geqslant 0$ and $e = 0$. We start with the case $\chi > 0$.

Proposition 10.3.36. *Every closed Seifert fibration with $\chi > 0$ and $e = 0$ is isomorphic to one of the following:*

$$S^2 \times S^1, \qquad \mathbb{RP}^2 \tilde{\times} S^1, \qquad (S^2, (p, q), (p, -q)).$$

The manifolds of the last type are all diffeomorphic to $S^2 \times S^1$.

Proof. If the base surface S is a sphere with $\leqslant 2$ singular fibres, we use Exercise 10.3.6. Otherwise S is one of the following orbifolds (see Table 6.1): $\cancel{(S^2, 2, 2, p)}$, $(S^2, 2, 3, 3)$, $(S^2, 2, 3, 4)$, $(S^2, 2, 3, 5)$, $\mathbb{RP}^2$, $\cancel{(\mathbb{RP}^2, p)}$ with $p \geqslant 2$. In all cases except $\mathbb{RP}^2$, we get $e \neq 0$ for any choice of Dehn filling parameters: for instance

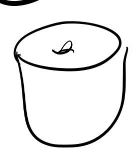
$$e(S^2, (2, q_1), (2, q_2), (p, q_3)) = \frac{q_1 + q_2}{2} + \frac{q_3}{p} \neq 0.$$

The other cases are analogous. □

The manifold $\mathbb{RP}^2 \tilde{\times} S^1$ is not diffeomorphic to $S^2 \times S^1$, because they have non-isomorphic fundamental groups. Moreover, $\mathbb{RP}^2 \tilde{\times} S^1$ is not prime:

$\chi^{orb}(B) = \chi(|B|) - \sum_i \left(1 - \frac{1}{p_i}\right)$

$2 \Rightarrow$ lens space



So now we want to produce $\Gamma < \mathrm{Isom}(S^2 \times \mathbb{R})$ s.t. $S^2 \times \mathbb{R} / \Gamma \cong S^2 \times S^1$ ① and $\mathbb{RP}^2 \tilde{\times} S^1$ ②

① $\Gamma = \langle (\mathrm{id}, \tau) \rangle$ where τ is a translation.

Any time we have base

② $\Gamma = \langle (i, r), (i, r') \rangle$ where i is the antipodal map and r, r' are reflections w.r.t. distinct pts in $\mathbb{R}$.

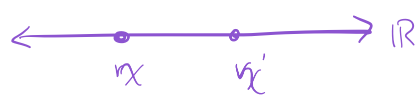
orbifold w/ one cone pt: (S, p)

$e(M) \neq 0$

Conversely, pick an orientable $M = S^2 \times \mathbb{R} / \Gamma$. The discrete group $\Gamma < \mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R})$ preserve the foliation into spheres $S^2 \times \{x\}$ which descend into a foliation by S^2 or $\mathbb{RP}^2$'s in M.

Therefore M decomposes into orient. int bundles $S^2 \times I$ and $\mathbb{RP}^2 \tilde{\times} I \Rightarrow M$ is either $S^2 \times S^1$ or $\mathbb{RP}^2 \tilde{\times} S^1$.

$$\begin{tikzcd} \mathbb{RP}^2 \tilde{\times} S^1 \arrow[r] & \mathbb{RP}^2 \\ S^2 \times S^1 \arrow[u] \arrow[r] & S^2 \arrow[u] \end{tikzcd}$$



$\mathbb{H}^2 \times \mathbb{R}$ case follows similar method:

12.4.2. $\mathbb{H}^2 \times \mathbb{R}$ **geometry.** We give $\mathbb{H}^2 \times \mathbb{R}$ the product metric. The discussion of the previous section applies as is to this case, showing that horizontal and vertical planes in the tangent spaces have sectional curvature -1 and 0. This in turn implies that

$$\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{R})$$

has four connected components, two being orientation-preserving. It is convenient to write the exact sequence

$$0 \longrightarrow \mathrm{Isom}(\mathbb{R}) \longrightarrow \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R}) \xrightarrow{p} \mathrm{Isom}(\mathbb{H}^2) \longrightarrow 0.$$

A discrete group $\Gamma < \mathrm{Isom}(X)$ is *cofinite* if $X/_\Gamma$ has finite volume.

Prop 12.4.4: A discrete group $\Gamma < \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ is cofinite iff both $p(\Gamma)$ and $\Gamma \cap \ker(p)$ are discrete and cofinite.

We will skip the proof even though it's cute.

Corollary 12.4.5: If the int(M) a compact orient. mfd admits a finite volume $\mathbb{H}^2 \times \mathbb{R}$ geometry then M is a Seifert mfd and $\chi < 0$. If M is closed then $e = 0$.

Pf: $\mathrm{Int}(M) = (\mathbb{H}^2 \times \mathbb{R})/\Gamma$ with Γ cofinite. By Prop. 12.4.4 the group $\Gamma \cap \ker(p)$ quotients any line $\{x\} \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$ to a circle in M, giving a Seifert fibration $M \to S$ onto the finite area orbifold $S = \mathbb{H}^2/p(\Gamma)$. We have $\chi(S) < 0$, and either $\partial M \neq 0$ or $e = 0$ because $\mathbb{H}^2 \times \{y\}$ projects to a section for $\underbrace{M \to S.}_{\text{kind of}}$

trivial bundle $= \Sigma \times S^1$

$$\begin{tikzcd} \tilde{M} \arrow[r, dashed] \arrow[d, dashed] & \Sigma \arrow[d] \\ M \arrow[r] & S \end{tikzcd}$$

$\tilde{M} \to \Sigma$ has a section $\Rightarrow \partial M \neq \emptyset$ or $e = 0$

$$\begin{tikzcd} \mathbb{H}^2 \times \{y\} \arrow[r, hook] \arrow[d] & \mathbb{H}^2 \times \mathbb{R} \arrow[d, "/\Gamma"] \\ \Sigma \arrow[r, dashed, "\exists"] & M \end{tikzcd}$$

This implies the same is true for M.

If $\exists\ M \xrightarrow{\pi} S$ where M is an S^1 bundle over a surface S. Then a section is $i: S \hookrightarrow M$

$\pi \circ i = id_S$.

We now prove the converse of Corollary 12.4.5.

Proposition 12.4.6. *If M is a Seifert manifold with $\chi < 0$ and either $\partial M \neq \varnothing$ or $e = 0$, the interior of M admits a finite-volume complete $\mathbb{H}^2 \times S^1$ geometry.*

Proof. By hypothesis there is a section Σ of $M \to S$, which is the fibre of a bundle $M \to O$ over a 1-orbifold O, see Section 11.4.4. The two structures give two exact sequences

$$0 \longrightarrow K \longrightarrow \pi_1(M) \xrightarrow{f} \pi_1(S) \longrightarrow 0,$$
$$0 \longrightarrow \pi_1(\Sigma) \longrightarrow \pi_1(M) \xrightarrow{g} \pi_1(O) \longrightarrow 0.$$

Since $\chi(S) < 0$ we may write $S = \mathbb{H}^2/_\Gamma$ and identify $\pi_1(S)$ with $\Gamma < \mathrm{Isom}(\mathbb{H}^2)$. Analogously we consider $\pi_1(O)$ inside $\mathrm{Isom}(\mathbb{R})$. The map

$$(f, g)\colon \pi_1(M) \longrightarrow \mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{R})$$

is injective and its image is discrete and acts freely on $\mathbb{H}^2 \times \mathbb{R}$, inducing a finite-volume $\mathbb{H}^2 \times \mathbb{R}$ structure on M. $\square$

$\Sigma \times S^1 = \tilde{M}$

$$\begin{tikzcd} \tilde{M} \arrow[r] \arrow[d] & \Sigma \arrow[d] \\ M \arrow[r] & S \end{tikzcd}$$

trivial bundle

$$\Rightarrow \exists \quad \begin{tikzcd} \tilde{M} \arrow[r] \arrow[d] & S^1 \arrow[d, dashed] \\ M \arrow[r, dashed] & O \end{tikzcd}$$