Last time: We showed that

Let ε_n be a Margulis constant. We can strengthen the Margulis Lemma.

Corollary 4.2.12. *Let $\Gamma < \text{Isom}(\mathbb{H}^n)$ be discrete and acting freely. For every point $x \in \mathbb{H}^n$ the subgroup $\Gamma_{\varepsilon_n}(x)$ is either trivial or elementary.* ✓

How? we showed that $\Gamma_{\varepsilon_n}(x)$ is virtually nilpotent so $\exists\ H \leq_{f.i.} \Gamma_{\varepsilon_n}(x)$

nilpotent $\Rightarrow$ nontrivial center $\Rightarrow$ H is trivial or $\underbrace{\text{elementary}}$

- gen by 1 hyp isom or
- gen by parabolics w/ same fixed pt.

Then this implies $\Gamma_{\varepsilon_n}(x)$ is trivial or elementary.

Thick-Thin decomposition:

We define a star-shaped set centred at $p \in \partial\mathbb{H}^n$ to be any subset $U \subset \mathbb{H}^n$ that intersects every half-line pointing to p in a half-line. For instance, a horoball is star-shaped. A *star-shaped neighbourhood* of a line $l \subset \mathbb{H}^n$ is any neighbourhood V of l that intersects every line orthogonal to l into a connected set. For instance, a R-neighbourhood of l is star-shaped.

These definitions pass to quotients. A *star-shaped cusp neighbourhood* is the quotient $U/_\Gamma$ of a Γ-invariant star-shaped set U centred at $p \in \partial\mathbb{H}^n$ via a discrete group Γ of parabolic transformations fixing p and acting freely. Analogously, a *star-shaped simple closed geodesic neighbourhood* is the quotient $V/_\Gamma$ of a Γ-invariant star-shaped neighbourhood V of l via a discrete group $\Gamma \cong \mathbb{Z}$ of hyperbolic transformations with axis l.

★ The truncated cusps and R-tubes studied in Section 4.1 are particularly nice star-shaped cusp and geodesic neighbourhoods.

Let $\varepsilon_n > 0$ be a Margulis constant. We define $M_{[\varepsilon_n,\infty)}$ and $M_{(0,\varepsilon_n]}$ respectively as the set of all points $x \in M$ having $\text{inj}_x M \geqslant \frac{\varepsilon_n}{2}$, and as the closure of the complementary set $M \setminus M_{[\varepsilon_n,\infty)}$. They form respectively the *thick* and *thin part* of M. ★★

The following theorem is arguably the most important structural result on complete hyperbolic manifolds of any dimension n.

Theorem 4.2.14 (Thick-thin decomposition). *Let M be a complete hyperbolic n-manifold. The thin part $M_{(0,\varepsilon_n]}$ consists of a disjoint union of star-shaped neighbourhoods of cusps and of simple closed geodesics of length $< \varepsilon_n$.*

truncated cusps $S \times [a,\infty)$

R-tubes

Proof. We have $M = \mathbb{H}^n/_\Gamma$. For every isometry $\varphi \in \Gamma$ we define

$$S_\varphi(\varepsilon) = \left\{ x \in \mathbb{H}^n \;\middle|\; d(\varphi(x), x) \leqslant \varepsilon \right\} \subset \mathbb{H}^n.$$

By Proposition 4.1.1 the thin part $M_{(0,\varepsilon_n]}$ is the image of the set

$$S = \left\{ x \in \mathbb{H}^n \;\middle|\; \exists \varphi \in \Gamma, \varphi \neq \text{id such that } d(\varphi(x), x) \leqslant \varepsilon_n \right\} = \bigcup_{\varphi \in \Gamma, \varphi \neq \text{id}} S_\varphi(\varepsilon_n).$$

Proposition 4.1.1. *Let $M = \mathbb{H}^n/_\Gamma$ be a complete hyperbolic manifold and $\pi \colon \mathbb{H}^n \to M$ the projection. For every $x \in M$ we have*

$$\text{inj}_x M = \frac{1}{2} \cdot d(\pi^{-1}(x)).$$

Suppose that $x \in S_\varphi(\varepsilon_n) \cap S_\psi(\varepsilon_n)$ for some non-trivial isometries $\varphi, \psi \in \Gamma$. By the Margulis Lemma both φ and ψ belong to the elementary group $\Gamma_{\varepsilon_n}(x)$ and hence by Proposition 4.2.9 both φ and ψ are either parabolic fixing the same point p at infinity or hyperbolic fixing the same line l.

Therefore every connected component S_0 of S is the union of all $S_\varphi(\varepsilon_n)$ where φ varies in some maximal elementary subgroup $\Gamma_0 < \Gamma$ of parabolics fixing the same point p or hyperbolics fixing the same line l. The set S_0 is a union of star-shaped sets centred at p or l and is hence also star-shaped.

The group Γ preserves S and the only isometries in Γ that preserve S_0 are those in Γ_0, therefore the quotient $M_{(0,\varepsilon_n]} = S/_\Gamma$ consists of star-shaped neighbourhoods of cusps and of simple closed geodesics. □

Corollary 4.2.16. *Let M be a complete hyperbolic n-manifold. The closed geodesics in M of length $< \varepsilon_n$ are simple and disjoint.*

Proof. These closed geodesics lie in the thin part. Star-shaped cusp neighbourhoods contain no closed geodesics, and each star-shaped geodesic neighbourhood contains only one closed geodesic, its core, which is simple. □

More on hyperbolic 3-mflds. Jump to:

CHAPTER 13

Mostow rigidity theorem

We have defined in Chapter 7 the Teichmüller space Teich(S_g) of a genus-g closed orientable surface S_g as the space of all the hyperbolic metrics on S_g, considered up to isometries isotopic to the identity; we have then proved that Teich(S_g) is homeomorphic to $\mathbb{R}^{6g-6}$ using the Fenchel–Nielsen coordinates.

This definition of Teich(M) actually applies to any closed hyperbolic manifold M, and we show here a striking difference between the dimensions two and three: if $\dim M = 3$ then Teich(M) is a single point. This strong result is known as the *Mostow rigidity Theorem*.

The impact of Mostow's rigidity on our understanding of three-dimensional topology cannot be overestimated. Thanks to this theorem every geometric information on a given closed hyperbolic three-manifold M like its volume, geodesic spectrum, etc. is promoted to a *topological* invariant of M, that is it depends on the differentiable structure of M only. In its strongest version, Mostow's rigidity says that the hyperbolic metric of M is fully determined by the group $\pi_1(M)$ alone.

Theorem 13.3.1 (Mostow rigidity). *Let M and N be two closed connected orientable hyperbolic 3-manifolds. Every isomorphism $\pi_1(M) \xrightarrow{\sim} \pi_1(N)$ between fundamental groups is induced by a unique isometry $M \xrightarrow{\sim} N$.*

This very powerful theorem says that an algebraic isomorphism between fundamental groups alone is enough to produce and characterise an isometry.

Corollary 13.3.2. *Two closed orientable hyperbolic 3-manifolds with isomorphic fundamental groups are isometric.*

Quick note on Sol manifolds:

Proposition 12.7.6. *The interior of a compact orientable manifold M admits a finite-volume complete* Sol *geometry if and only if it is a torus (semi-)bundle of Anosov type.*

So we have:

Proposition 12.9.7. *Let M_A be a torus bundle with monodromy $A \neq \pm I$. The following holds:*

- if $|\mathrm{tr}A| < 2$, *i.e.* A has finite order, then M_A is flat;
- if $|\mathrm{tr}A| = 2$, *i.e.* A is reducible, then M_A is Nil;
- if $|\mathrm{tr}A| > 2$, *i.e.* A is Anosov, then M_A is Sol.

Geometrization!!

12.9.1. Statement and main consequences. We say that a compact 3-manifold with (possibly empty) boundary consisting of tori is *geometric* if its interior has a finite-volume complete geometric structure modelled on one of the eight geometries:

$$S^3,\ \mathbb{R}^3,\ \mathbb{H}^3,\ S^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ \mathrm{Nil},\ \mathrm{Sol},\ \widetilde{\mathrm{SL}_2}.$$

The following conjecture was formulated by Thurston in 1982:

Conjecture 12.9.1 (Geometrisation Conjecture). *Let M be an irreducible orientable compact 3-manifold with (possibly empty) boundary consisting of tori. Every block of the geometric decomposition of M is geometric.*

The conjecture has been proved by Perelman in 2002 and its proof goes very very far from the scope of this book. It is however quite easy to deduce important consequences from it.

Conjecture 12.9.2 (Poincaré conjecture). *Every simply connected closed 3-manifold M is diffeomorphic to S^3.*

Proof using geometrisation. Via the prime decomposition we may restrict to the case M is prime, hence irreducible. The group $\pi_1(M)$ is trivial and hence does not contain $\mathbb{Z}\times\mathbb{Z}$: every torus in M is thus compressible and the geometric decomposition is trivial. By geometrisation M is itself geometric. The only geometry with finite fundamental groups is S^3, and hence $M = S^3/_\Gamma$ is elliptic. Since M is simply connected, the group $\Gamma = \pi_1(M)$ is trivial and hence $M = S^3$. $\square$

Conjecture 12.9.4 (Hyperbolisation). *Every closed irreducible 3-manifold M with infinite $\pi_1(M)$ not containing $\mathbb{Z} \times \mathbb{Z}$ is hyperbolic.*

Proof using geometrisation. Since $\pi_1(M)$ does not contain $\mathbb{Z} \times \mathbb{Z}$ every torus is compressible and the geometric decomposition of M is trivial. By geometrisation M is geometric. Its geometry is not S^3 since $\pi_1(M)$ is infinite, and is not $S^2 \times \mathbb{R}$ since M is irreducible. In the other Seifert geometries and in Sol the fundamental group $\pi_1(M)$ always contain a $\mathbb{Z} \times \mathbb{Z}$ (there is always a finite covering containing an incompressible torus). □

Just so you know:

Ch. 13 covers proof of Mostow Rigidity

Ch. 14 Thurston's equations and constructing hyp. mfds via triangulations

In dimension two every closed hyperbolic surface is constructed by gluing some geodesic pair-of-pants. In dimension three, although closed hyperbolic three-manifolds are everywhere, it is somehow harder to construct them explicitly: the most general procedure to determine a hyperbolic metric (if any) on a given closed 3-manifold consists of solving *Thurston's equations*.

Thurston's equations arise naturally in the attempt of constructing a hyperbolic three-manifold by triangulating it into hyperbolic tetrahedra. The

Ch. 15 Hyperbolic Dehn Filling Thm (generically filling a cusped hyp. 3-mfd results in a hyp 3-mfd)

Corollary 15.1.3. *If M is a complete orientable finite-volume hyperbolic three-manifold with one cusp, all but finitely many Dehn fillings M^{fill} are hyperbolic.*