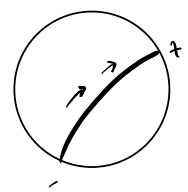
Ch. 3 Hyperbolic Geometry

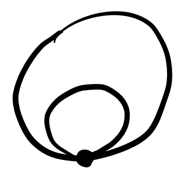
Definition 3.1.1. A *hyperbolic* (resp. *flat* or *elliptic*) *manifold* is a connected Riemannian n-manifold that may be covered by open sets isometric to open sets of $\mathbb{H}^n$ (resp. $\mathbb{R}^n$ o S^n).

Theorem 3.1.2. *Every complete simply connected hyperbolic n-manifold M is isometric to $\mathbb{H}^n$.*

Proposition 3.1.3. *Every complete hyperbolic n-manifold M is isometric to $\mathbb{H}^n/_\Gamma$ for some subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ acting freely and properly discontinuously.*

Remark 3.1.4. A group $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ acts freely if and only if it does not contain elliptic isometries: that is, every non-trivial isometry in Γ is either hyperbolic or parabolic.

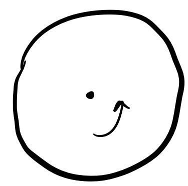




Corollary 3.1.6. *There is a natural 1-1 correspondence*

$$\left\{\begin{array}{c}\text{complete hyperbolic}\\ \text{manifolds } M \text{ up to isometry}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{discrete subgroups } \Gamma < \mathrm{Isom}(\mathbb{H}^n)\\ \text{without elliptics}\\ \text{up to conjugation}\end{array}\right\}$$

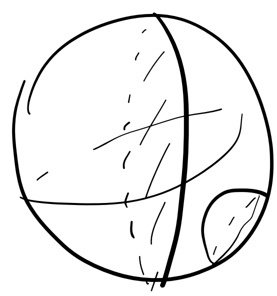
Proof. When passing from the complete hyperbolic manifold M to the group Γ, the only choice we made is an isometry between the universal cover of M and $\mathbb{H}^n$. Different choices produce conjugate groups Γ. □



3.1.4. Coverings. We now make a simple but crucial observation: if $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ acts freely and properly discontinuously, then also every subgroup $\Gamma' < \Gamma$ does; we get a manifolds covering

$$\mathbb{H}^n/_{\Gamma'} \longrightarrow \mathbb{H}^n/_\Gamma$$

whose degree d is precisely the index of Γ' in Γ. Recall from Proposition 1.2.20

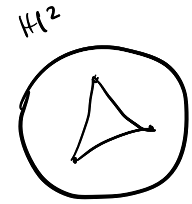


3.2. Polyhedra

A polyhedron in $\mathbb{H}^n$ is a natural geometric object, that may be used to visualise discrete groups in $\mathrm{Isom}(\mathbb{H}^n)$ and hence hyperbolic manifolds. Polyhedra may sometimes be combined to form some tessellations of the space.

3.2.1. Polyhedra. A *half-space* in $\mathbb{H}^n$ is the closure of one of the two portions of space delimited by a hyperplane. We say that a set of half-spaces is locally finite if their boundary hyperplanes are.

Definition 3.2.1. A n-dimensional *polyhedron* P in $\mathbb{H}^n$ is the intersection of a locally finite set of half-spaces. We also assume that P has non-empty interior.



The *convex hull* of a set $S \subset \overline{\mathbb{H}^n}$ is the intersection of all the convex sets containing S.

Exercise 3.2.4. The convex hull of finitely many points in $\mathbb{H}^n$ that are not contained in a hyperplane is a compact polyhedron. Conversely, every compact polyhedron has finitely many vertices and is the convex hull of them.

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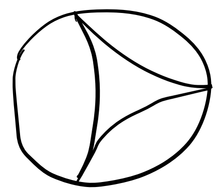
3.2.2. Finite polyhedra. We now enlarge slightly the class of compact polyhedra by admitting finitely many vertices at infinity.

Definition 3.2.5. A *finite polyhedron* is the convex hull of finitely many points $x_1, \ldots, x_k \in \overline{\mathbb{H}^n}$ that are not contained in the closure of a hyperplane.

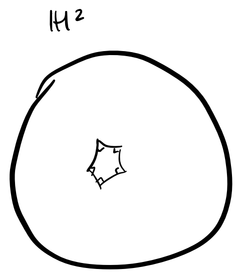
The x_i's that lie in $\partial\mathbb{H}^n$ are called *ideal vertices*, while the usual vertices of P are the *finite* or *actual vertices*. The ideal vertices form the set $\overline{P} \setminus P$.



Proposition 3.2.8. *Every finite polyhedron has finite volume.*

Proof. For every ideal vertex of P, a small horoball centred at p intersects P into a cone that has finite volume. The polyhedron P decomposes into finitely many cones and a bounded region, see Figure 3.1-(right). □

A finite polyhedron without finite vertices is called an *ideal polyhedron*.



3.3. Tessellations

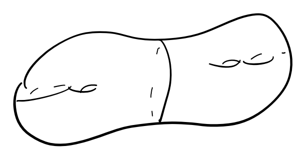
A tessellation is a nice paving of $\mathbb{H}^n$ made of polyhedra. Not only tessellations are beautiful objects, but they are also tightly connected with discrete groups of $\mathrm{Isom}(\mathbb{H}^n)$ and hence with hyperbolic manifolds.

Definition 3.3.1. A *tessellation* of $\mathbb{H}^n$ (or $\mathbb{R}^n$, S^n) is a locally finite set of polyhedra that cover the space and may intersect only in common faces.

every $\pi_1(S) \hookrightarrow \Gamma_{⬠}$

S hyp.

3.3.2. Platonic solids. We now turn to tessellations of 3-dimensional spaces. Table 3.1 displays a finite list of platonic solids P with dihedral angles $\frac{2\pi}{k}$ contained in $\mathbb{H}^3$, $\mathbb{R}^3$, or S^3. For each solid in the list, by reflecting iteratively P along its faces we get a tessellation of the space.



polyhedron	$\theta = \frac{\pi}{3}$	$\theta = \frac{2\pi}{5}$	$\theta = \frac{\pi}{2}$	$\theta = \frac{2\pi}{3}$
tetrahedron	ideal $\mathbb{H}^3$	S^3	S^3	S^3
cube	ideal $\mathbb{H}^3$	$\mathbb{H}^3$	$\mathbb{R}^3$	S^3
octahedron			ideal $\mathbb{H}^3$	S^3
icosahedron				$\mathbb{H}^3$
dodecahedron	ideal $\mathbb{H}^3$	$\mathbb{H}^3$	$\mathbb{H}^3$	S^3

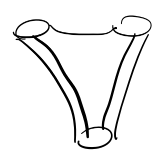
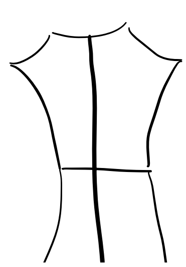
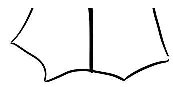


Table 3.1. The platonic solids with dihedral angle θ that divide 2π.





3.4.6. Coxeter polyhedra. Triangle groups may be generalised to all dimensions as follows. A polyhedron P in $\mathbb{H}^n$ (or $\mathbb{R}^n$, S^n) is a *Coxeter polyhedron* if the dihedral angle of every codimension-two face divides π. For instance, the regular ideal tetrahedron and octahedron are Coxeter polyhedra.

The following theorem generalises Proposition 3.4.6. Let P be a finite Coxeter polyhedron: it is the convex hull of finitely many vertices in $\overline{\mathbb{H}^n}$ (or $\mathbb{R}^n$, S^n) and has k facets, that we number as $1, \ldots, k$; we denote by r_i the reflection along the i-th facet, and by $\frac{\pi}{a_{ij}}$ the dihedral angle formed by the i-th and j-th facets, if they are incident. Let Γ be the isometry group generated by the reflections along the facets of P.

Theorem 3.4.7. *By mirroring iteratively a finite Coxeter polyhedron P along its facets we get a tessellation of $\mathbb{H}^n$ (or $\mathbb{R}^n, S^n$). The group Γ acts freely and transitively on the tessellation: hence it is discrete and P is a fundamental domain for Γ. A presentation for Γ is*

$$\langle r_1, \ldots, r_k \mid r_i^2, (r_i r_j)^{a_{ij}} \rangle$$

where i varies in $1, \ldots, k$ and the pair i, j varies among the incident facets.

$$\langle r_1, \ldots, r_k \mid (r_i r_j)^2 \rangle \quad \text{v.a.A.g.}$$

$$\approx [r_i, r_j]$$

A group generated by some reflections along hyperplanes is called a *reflection group.* The following proposition shows that Coxeter polyhedra generate all the interesting reflection groups.

Proposition 3.4.8. *Every discrete reflection group Γ is generated by the reflections along the facets of some Coxeter polyhedron.*

Coxeter polyhedra are beautiful objects that can be used to construct hyperbolic manifolds: every finite Coxeter polyhedron P generates a reflection group Γ which contains, by Selberg's Lemma, a torsion-free subgroup Γ' of some finite index h. The quotient $M = \mathbb{H}^n/_{\Gamma'}$ is a hyperbolic manifold and is tessellated into h copies of P, so that $\mathrm{Vol}(M) = h\mathrm{Vol}(P)$. By residual finiteness, there are plenty of such manifolds.

3.5.1. Hyperbolic manifolds with geodesic boundary. We reformulate a definition of hyperbolic (elliptic, flat) manifolds that allows the presence of some geodesic boundary. These manifolds are useful because they can be glued along their boundaries to produce new hyperbolic (elliptic, flat) manifolds.

Definition 3.5.1. A *hyperbolic* (*elliptic, flat*) *manifold M with geodesic boundary* is a Riemannian manifold with boundary where every point has an open neighbourhood isometric to an open set in a half-space in $\mathbb{H}^n$ (S^n, $\mathbb{R}^n$).

The boundary ∂M of a hyperbolic (elliptic, flat) n-manifold with geodesic boundary is a hyperbolic (elliptic, flat) $(n-1)$-manifold without boundary. Theorem 3.1.2 extends appropriately to this context.

Theorem 3.5.2. *Every complete simply connected hyperbolic n-manifold M with geodesic boundary is isometric to the intersection of some half-spaces in $\mathbb{H}^n$ with disjoint boundaries.*

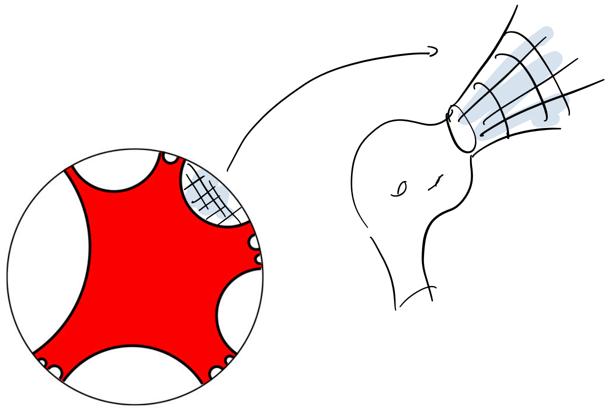


Figure 3.17. An intersection of (possibly infinitely many!) half-planes. The universal cover of a hyperbolic surface with boundary is isometric to such an object.

3.5.2. Cut and paste. Hyperbolic manifolds with geodesic boundary are useful because they can be glued to produce new hyperbolic manifolds.

Let M and N be hyperbolic manifolds with geodesic boundary and $\psi\colon \partial M \to \partial N$ be an isometry. Let $M \cup_\psi N$ be the topological space obtained by quotienting the disjoint union $M \sqcup N$ by the equivalence relation that identifies p to $\psi(p)$ for all $p \in \partial M$.

Proposition 3.5.4. *The space $M \cup_\psi N$ has a natural structure of hyperbolic manifold.*