Last class we were studying $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL_2}$ mflds.

Both have $\chi(B) < 0$ and we characterized these according to their scaffolding A. If $A(m)$ was scaffolding w/ $m \geq 5$ we called B generic. If $m = 3, 4$ then B is called exceptional.

If $m < 3$, then $\chi(B)$ cannot be negative.

Then we constructed Y the branched scaffolding that admits immersions of all $A(m)$ where $m \geq 5$. In fact, we will punch an extra hole into the scaffolding for all B (we explain the reason below) and so the Y below actually admits immersions of all $A(m)$ with $m \geq 6$.

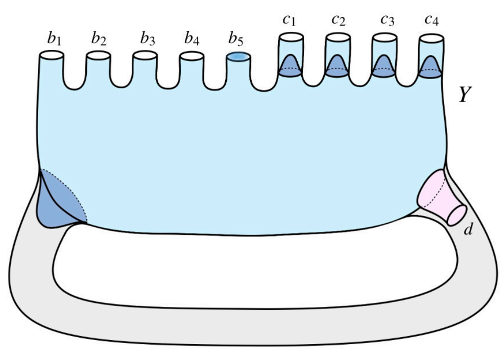


Figure 2: The branched scaffolding Y

Aim: merge all boundary comps. of Y except d to obtain a branched surface X (see below). Then consider $X \times S^1$ and attach a "swiss army knife" of bundle attachments which are S^1 bundles over the scaffolding attachments (Möb, A, disc w/ cone pt.)

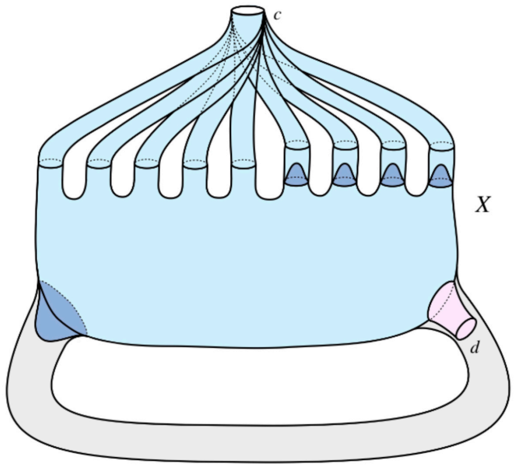


Figure 4: Merging of the $\mathbb{A}$

A way to envision $X \times S^1$:

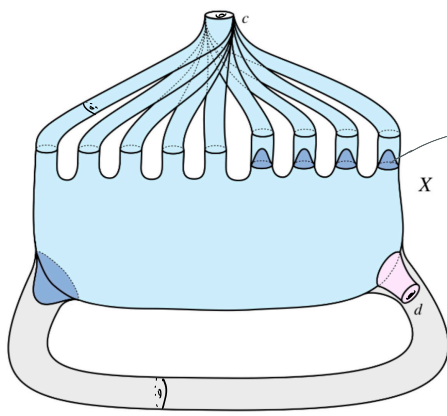


Figure 4: Merging of the $\mathbb{A}$

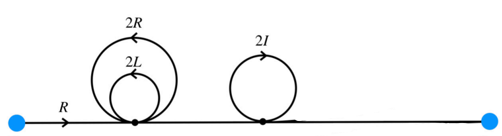
$\mathbb{D} \times S^1$ = solid torus

only orient S^1 bundle over Möb

only orient. S^1 bundle over annulus $\mathbb{A}$.

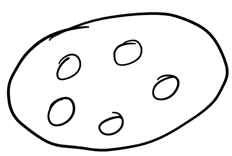
Now at $c \times S^1$ we attach Möb $\tilde{\times}\, S^1$, $\mathbb{A} \times S^1 = T^2 \times I$, and all possible S^1 bundles over a disc w/ a cone pt of any order

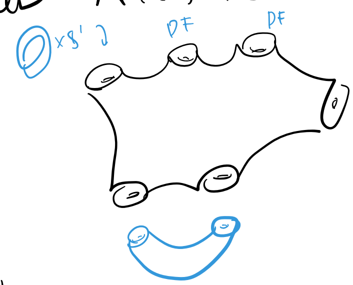
—— use tool we saw for spherical mfds!



The result is a branched mfd W that admits immersions of all orient. SF mfds M w/ exactly one bdy comp and B a generic orbifold w/ $\chi(B)<0$.

From this branched mfd we then want to differentiate between $\mathbb{H}^2\times\mathbb{R}$ and $\widetilde{SL}_2$ mfds by either ensuring $e=0$ or $e\neq 0$ resp. So we do a Dehn filling that either extends product structure or destroys it. Unfortunately, our setup makes this impossible. To understand why, consider the following two mfds:

Take $A(6) =$  and consider $A(6)\times S^1$



At 2 of the boundary tori, attach $T^2\times I$

At 1 " ", attach $\text{Möb}\,\tilde{\times}\,S^1$

At 1 " " do a Dehn filling of slope $\frac{1}{3}$ to obtain cone pt in base B of order 3

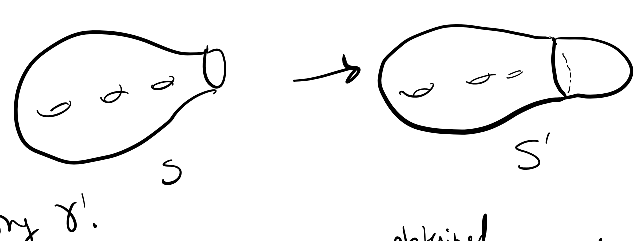
At 1 " " do a Dehn filling of slope $\frac{1}{4}$ to obtain a cone pt in B of order 4

The result is an orient. SF 3-mfd M w/ exactly 1 boundary torus. Recall that a SF mfd w/ boundary <u>always</u> has a horiz. surface.

Now construct M' in <u>almost</u> the same way <u>except</u> do Dehn fillings with slopes $\frac{2}{3}$ and $\frac{3}{4}$.

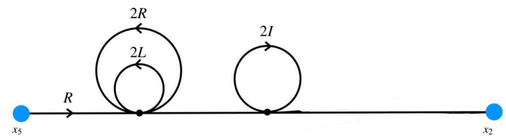
M and M' both immerse in the branched mfd W.
Each contain a horizontal surface (transvere to fiber)
<u>BUT</u> the boundary of such a surface in ∂M vs. $\partial M'$ will correspond to two different curves γ and γ', resp., in $\partial W = d \times S^1$.

Now consider the mfds. $\overline{M}$ and $\overline{M}'$ obtained from M, M' by extending the product structure so that $\overline{M}$ and $\overline{M}'$ are $\mathbb{H}^2 \times \mathbb{R}$ mfds. In particular, $\overline{M}$ is obtained from M by Dehn filling along γ so that horiz. surface S in M extends to a horiz surface $\overline{S}$ in $\overline{M}$:



Similarly, $\partial \overline{M}'$ is filled along γ'.
<u>Issue</u>: If we want a branched mfd $M_{\mathbb{H}^2 \times \mathbb{R}}$ obtained from W, admitting immersions of all $\mathbb{H}^2 \times \mathbb{R}$ mfds w/ generic base B, we would need to Dehn fill @ ∂W along diff curves for diff. mfds M.

Solution: replace



with a tool called a variable branched disc so that the boundary of a horizontal surface in any mfd M admitting an immersion to W is always the same prescribed curve in ∂W. In this way we change our branched mfd W to "pick out" the horizontal surface in a canonical way. Call this curve that is the canonical bdry of a horiz surface α in ∂W.

Then to construct $M_{\mathbb{H}^2 \times \mathbb{R}}$ from W we simply Dehn fill along α.

To construct $M_{\widetilde{SL_2}}$ we allow all other Dehn fillings at ∂W except along α (this destroys product structure and ensures $e \neq 0$).

We handle the exceptional cases similarly according to the following chart of possible scaffolding attachments.

Exceptional Case	$\partial 1$ **attachments**	$\partial 2$ **attachments**	$\partial 3$ **attachments**	$\partial 4$ **attachments**
$(4, \leq 3)$	Möb, $\mathbb{A}$, $p_1 \geq 2$	Möb, $\mathbb{A}$, $p_2 \geq 2$	Möb, $\mathbb{A}$, $p_3 \geq 2$	Möb, $\mathbb{A}$,
$(4, 4; \geq 3)$	$p_1 \geq 3$	$p_2 \geq 2$	$p_3 \geq 2$	$p_4 \geq 2$
$(3, 3; \geq 7, \geq 3)$	$p_1 \geq 7$	$p_2 \geq 3$	$p_3 \geq 2$	$\times$
$(3, 3; \geq 5, \geq 4)$	$p_1 \geq 5$	$p_2 \geq 4$	$p_3 \geq 2$	$\times$
$(3, 3; \geq 4, \geq 3, \geq 3)$	$p_1 \geq 4$	$p_2 \geq 3$	$p_3 \geq 3$	$\times$
$(3, 2; \geq 3)$	$p_1 \geq 3$	$p_2 \geq 2$	Möb,	$\times$
$(3, \leq 1)$	Möb, $\mathbb{A}$, $p_1 \geq 2$	Möb, $\mathbb{A}$,	Möb, $\mathbb{A}$,	$\times$