A Combinatorial Characterization of Geometric 3-manifolds

Joint w/ D. Cooper and L. Mavrakis

Overall idea (Thurston):

Let $\mathcal{J}$ be a family of manifolds

find $\mathcal{M}$, a finite construction kit with gluing rules

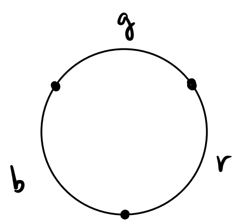
such that

$\mathcal{G}(\mathcal{M}) \triangleq$ the set of (cpt) manifolds that can be built with the pieces of $\mathcal{M}$

is exactly $\mathcal{J}$.

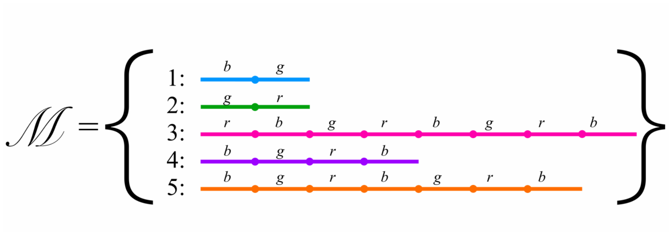
Example:

$S^1 =$



, let $\mathcal{J} = \{$covers of S^1 of deg $\geq 3\}$

Claim:



with the following gluing rules build exactly $\mathcal{J}$

Gluing Rules: ($i \to j$ means that the end of i is glued to start of j)

$1 \to 2 \quad 2 \to 3 \quad 3 \to 4 \quad 3 \to 5 \quad 3 \to 1 \quad 4 \to 1 \quad 5 \to 1$

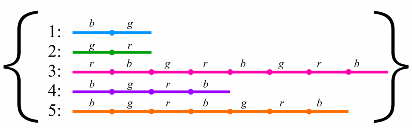
Pf sketch: 2 parts

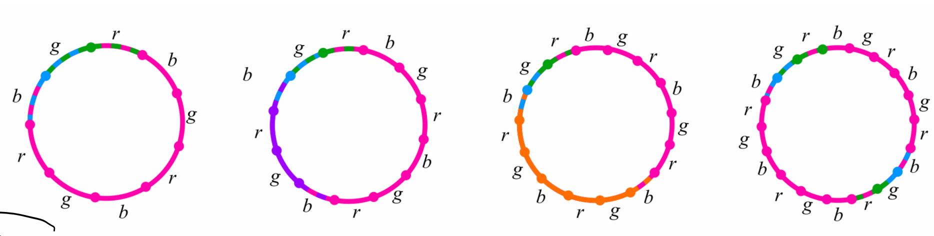
1. $\mathcal{M}$ builds all covers of deg ≥ 3
2. $\mathcal{M}$ builds no other closed 1-mfds.

1. **Note 1:** pieces 1, 2, 3 build all covers of deg $3k$

   Note 2: pieces 1, 2, 3, 4 build all covers of deg $3k+1$, $3k$

   Note 3: pieces 1, 2, 3, 5 build all covers of deg $3k+2$, $3k$

$\mathcal{M} =$ 



$1 \to 2 \to 3$

deg 3

deg 4

$1 \to 2 \to 3 \to 4$

deg 5

deg 6

deg 7 $\quad 1 \to 2 \to 3 \to 1 \to 2 \to 3 \to 4$

② How do we avoid building any other closed 1-mfds?

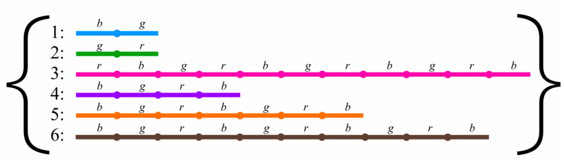
(i) leave out piece 

(ii) choose gluing rules carefully

so that

all closed mfds built with $\mathcal{M}$ must contain a 3-piece.

Claim: Let $\mathcal{S} = \{$ covers of S^1 of deg $\geq 4 \}$.

Then $\mathcal{N} =$  is a Lego set for $\mathcal{S}$.

Def: A <u>local model</u> is (K, v) where $K =$ triangulation of $\mathbb{B}^n$, $v \in K^\circ$

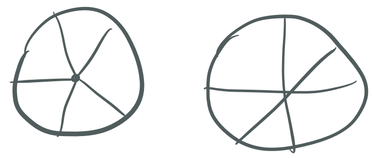
Let $\mathcal{M}$ be a <u>finite</u> set of local models.

A simplicial complex L is <u>modelled on $\mathcal{M}$</u> if for each vertex w of L, $\exists$ a nbd. of w that is s. homeo to a $(K, v) \in \mathcal{M}$.

a family of manifolds is locally combinatorially defined (LCD) if $\mathcal{F} = \mathcal{G}(\mathcal{M})$ for some $\mathcal{M}$.

Examples of LCD families :

1) Covers of $\mathbb{S}^1$ of deg $\geq k$



2) Covers of $\mathbb{S}^1$

3) Let $\mathcal{M}_5 = \{ (D_5, v) \}$
$\mathcal{M}_6 = \{ (D_6, v) \}$
$\mathcal{M}_7 = \{ (D_7, v) \}$

where D_n is a disc formed by cyclically arranging n triangles around v

$$\mathcal{G}(\mathcal{M}_n) = \text{set of compact surfaces with } \chi(S) \begin{cases} > 0 & n=5 \\ = 0 & n=6 \\ < 0 & n=7 \end{cases}$$

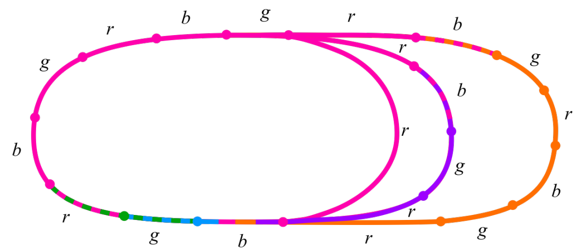
Note: $\mathcal{M} = \{ (D_5, v), (D_6, v), (D_7, v) \}$ then
$\mathcal{G}(\mathcal{M}) = \{$ all cpt surfaces $\}$

4.) Theorem (Cooper-Thurston, Walker '85) : The set of closed orientable 3-mfds is LCD using 3 local models.

Main Theorem (Cooper-Mavrakis-P.):
The set of closed orientable 3-manifolds that admit a particular Thurston geometry is LCD. (8 theorems in one)

We need an alternate perspective :

Claim: The set of closed 1-mfds that immerses in

$W =$ 

is exactly the set of covers of $\mathbb{S}^1$ of deg ≥ 3.

① all of these covers immerse in W

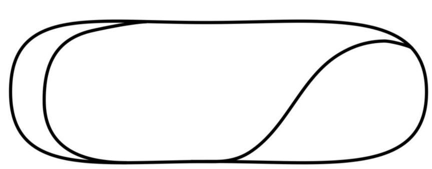
② a closed mfd that immerses in W is one of these covers

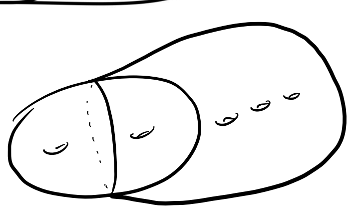
EQUIVALENCE

a family $\mathcal{J}$ of n-mfds is LCD. $\Longleftrightarrow$ $\exists$ branched n-mfd W s.t. $M \hookrightarrow W$ iff $M \in \mathcal{J}$

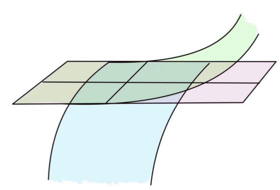
Loose Def: A branched n-manifold is a generalization of a train track and branched surface

- Dim 1 = train track

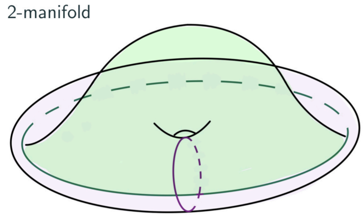




- Local picture in dimension 2



- A branched 2-manifold



Thurston's 8 geometries:

$\boxed{\mathbb{S}^2 \times \mathbb{R},\ \mathbb{S}^3,\ \mathbb{H}^2 \times \mathbb{R},\ \widetilde{SL}(2,\mathbb{R}),\ \mathbb{E}^3,\ \mathrm{Nil}}$ Hyperbolic Sol these are all Seifert fibered spaces (SFS), i.e. a circle bundle over a 2-dim orbifold B.

Today: Focus on spherical manifolds.

Def: A spherical manifold $M = \mathbb{S}^3/\Gamma$ where $\Gamma < SO(4)$ is a finite subgroup acting freely by rotations on $\mathbb{S}^3$ (all such mfds are closed, orientable)

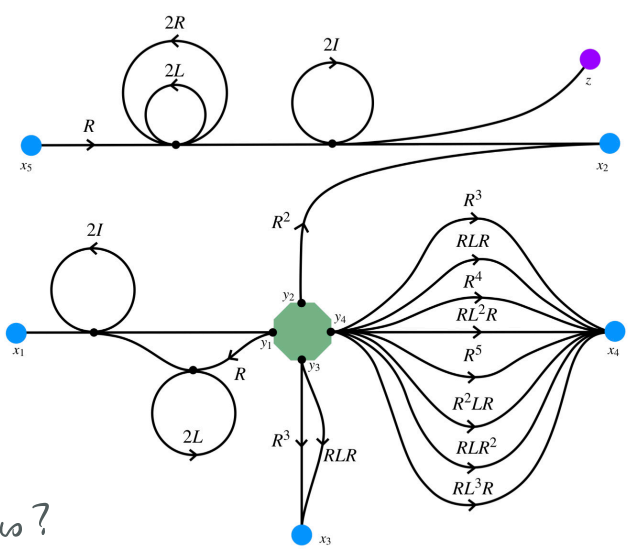
But also, a SFS admits a Thurston geometry determined by $e(M) \in \mathbb{Q}$ and $\chi^{orb}(B)$. e(M) measures the defect from M being a product.

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{E}^2 \times \mathbb{R}$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	$\mathbb{S}^3$	Nil	$\widetilde{SL_2}$

Equiv. Def: A spherical manifold is a SFS w/ $\chi^{orb}(B) > 0$ and $e(M) \neq 0$.

Γ representing branched 3-mfd M_s

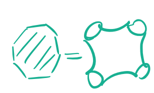
$\exists\ \pi : M_s \to \Gamma$



every point on black edge rep. T^2

• = solid torus

• = $K \tilde{\times} I$

 $\times\ \mathbb{S}^1$

Where are the lens spaces?

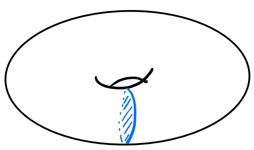
Theorem (CMP): A closed orientable 3-mfd M immerses in M_5 iff M is a spherical manifold.

Types of spherical 3-manifolds:
Possible base orbifolds B with $\chi^{orb}(B) > 0$

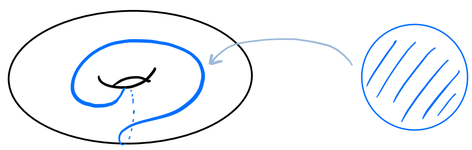
1) $S^2(2,2,n)$ $n>0$
2) $\mathbb{RP}^2(n)$ $n>0$ } lens spaces
3) $S^2(n,n)$ $n>0$ } prism manifolds
4) $S^2(2,3,n)$ $n=3,4,5$ } other

Terminology: Let $A = \mathbb{D}^2 \times S^1$ be a solid torus. The curve α in $\partial A \cong T^2$ bounding the disc $\mathbb{D}^2$ is called the <u>meridian</u> of A.

Exs:



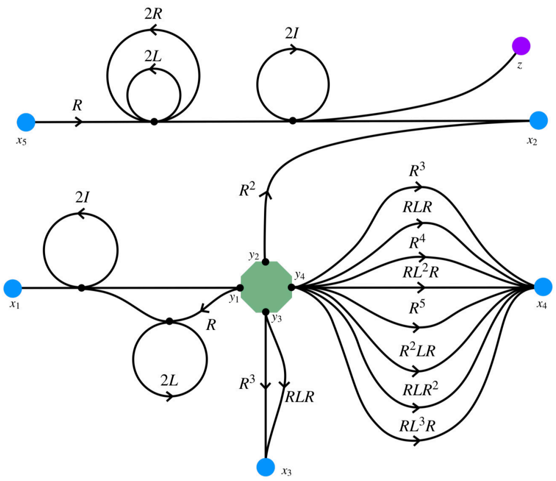
Standard solid torus
standard meridian



Def: A <u>lens space</u> M is obtained from two solid tori A and C by gluing ∂A to ∂C subject to the condition that the meridian α of A does not equal the meridian γ of C.

Note: Fixing A to be the standard solid torus, then in the basis $\binom{1}{0}, \binom{0}{1}$ for $H_1(\partial C, \mathbb{Z})$, $\gamma = \binom{p}{q}$ where p, q are coprime and $\binom{p}{q} \neq \binom{1}{0}$.

Question: where do lens spaces possibly immerse in



?

Note: Solid torus $\cup$ $T^2 \times I$ = solid torus

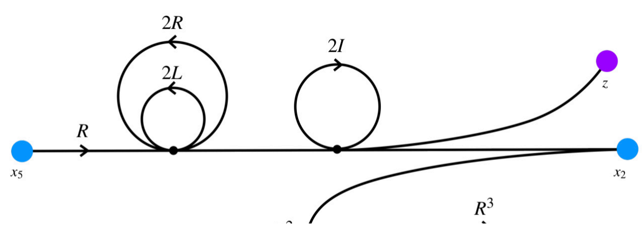
Consider matrices $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then,

$$L\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix} \qquad R\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a+b \end{pmatrix}$$

EX: Suppose A, C are solid tori. $\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \partial A$ (standard merid.)
and $\gamma = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \partial C$

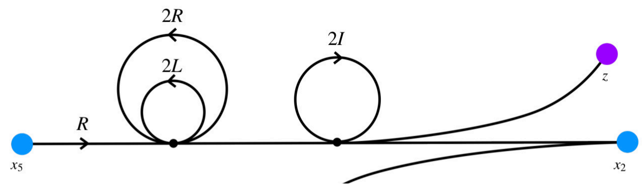
C  A

Then $R \cdot R \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.



Lemma: For any $\binom{p}{q} \neq \binom{1}{0}$ with p, q coprime
$\exists\ Q \in \langle L, R\rangle \cdot R$ s.t. $Q\binom{1}{0} = \binom{p}{q}$.

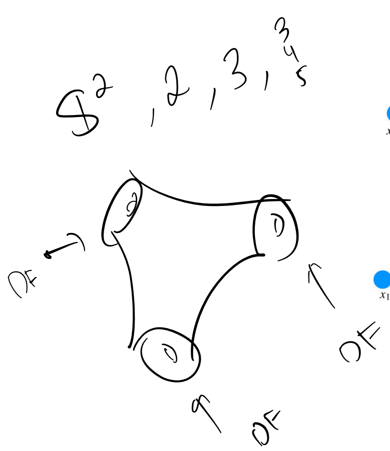
Therefore, any lens space immerses in M_5 with immersion path endpts $x_2, x_5 \in \Gamma$.

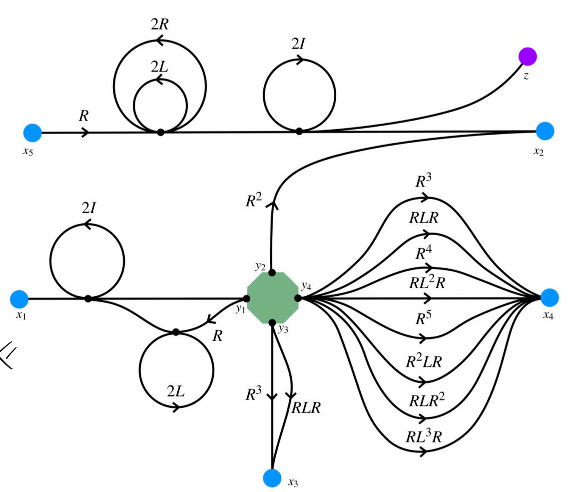


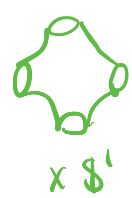
What about the other direction of the proof?

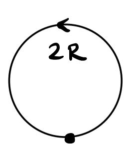
We have taken precautions s.t. $\mathbb{S}^2 \times \mathbb{S}^1$ and $T^2 \times I/\sim$ don't immerse in M_5

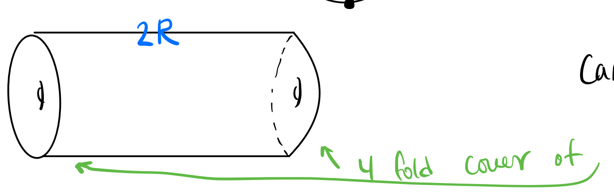
Lemma: For any closed mfd. M that immerses in M_5, the immersion path in Γ has endpoints in $\{x_1, x_2, \ldots, x_5, z\}$.





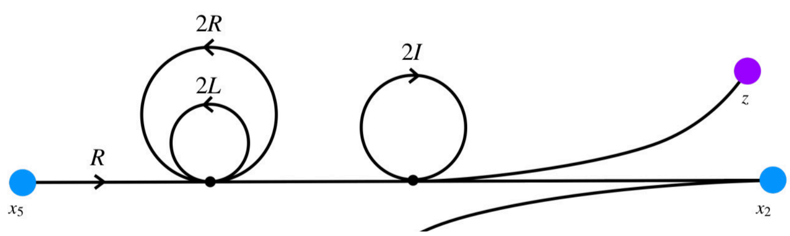


Note: $T^2 \times I$ ↷  is impossible



Can't glue up and get a mfd.

Lemma: If M is closed mfd. w/ immersion path endpts X_2 and X_5, then M is a lens space
(or a cover of a lens space)



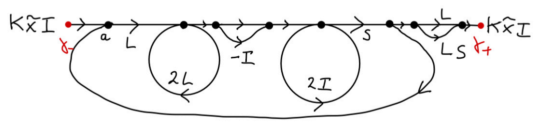


Figure 10: A branched manifold for SOL