Basics of Geometry of $\mathbb{H}^3$:

$\mathbb{H}^3 = \{(x,y,u) \in \mathbb{R}^3 : u>0\}$

$\gamma(t)$ a p.wise diff. curve in $\mathbb{H}^3$, $\gamma(t) = (x(t), y(t), u(t))$
its hyperbolic length is

$$l_{hyp}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2 + u'(t)^2}}{u(t)} \, dt$$

metric is $(ds)^2 = \dfrac{dx^2 + dy^2 + du^2}{u^2}$

$\mathbb{H}^3$ admits several "obvious" isometries:

horizontal translations: $\varphi(x,y,u) = (x+x_0, y+y_0, u)$
$x_0, y_0 \in \mathbb{R}$

homotheties/scaling: $\varphi(x,y,u) = (\lambda x, \lambda y, \lambda u)$
for $\lambda \in \mathbb{R}$

and rotations around u-axis defined by

$$\varphi(x,y,u) = (\underbrace{x\cos\theta - y\sin\theta}_{x'}, \underbrace{x\sin\theta + y\cos\theta}_{y'}, u)$$

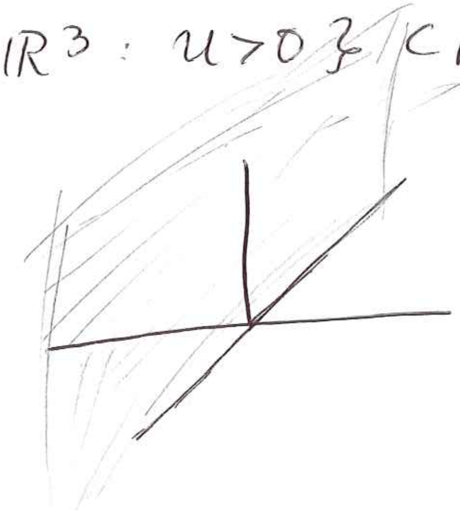
for $\theta \in \mathbb{R}$

Coming from rotation of $\mathbb{C}$ within $\mathbb{H}^3$: $e^{i\theta}z =$
$(\cos\theta + i\sin\theta)(x+iy) \rightsquigarrow x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta)$
$\mathrm{Re}(\cdot) = x'$ $\quad \mathrm{Im}(\cdot) = y'$

Analog of inversion across unit circle is inversion across unit sphere $\varphi: \mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$

$$\varphi(x,y,u) = \left(\frac{x}{x^2+y^2+u^2}, \frac{y}{x^2+y^2+u^2}, \frac{u}{x^2+y^2+u^2} \right)$$

Setting $y=0$ $H = \{(x,0,u) \in \mathbb{R}^3 : u>0\} \subset \mathbb{H}^3$ has natural ident. w/ $\mathbb{H}^2$.



Think of $\mathbb{C} \subset \mathbb{H}^3$ as $\{(x,y,0)\}$ (xy-plane)

Isometries:

Every linear or antilinear fractional trans. of $\hat{\mathbb{C}}$ continuously extends to a map $\hat{\varphi}: \mathbb{H}^3 \cup \hat{\mathbb{C}} \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ whose restriction to $\mathbb{H}^3$ is an isometry.

Pf: any such map is a composition of trans, rotations, scaling and inversions across unit circle which all extend to isoms of $\mathbb{H}^3$ as we stated above.

Conversely, every isom. of $\mathbb{H}^3$ is obtained as an extension of a linear or antilinear frac. trans.

Pf: P. 232 - 233 Bonahon

Note: A good way to think about ~~this~~ is that the inversions about circle extend to inversions about orthogonal hemisphere whose boundary is that circle in $\hat{\mathbb{C}}$.

Classification of Orientation Preserving Isoms of $\mathbb{H}^3$:

Now that we know every orientation preserving isom. of $\mathbb{H}^3$ arises as an extension of a linear fractional trans. of $\hat{\mathbb{C}}$. $\varphi(z) = \frac{az+b}{cz+d}$ $a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$.
i.e. $ad-bc=1$.

$\varphi(z)$ fixes exactly one or two points of $\hat{\mathbb{C}}$.

$$z = \frac{az+b}{cz+d} \implies cz^2 + (d-a)z - b = 0$$

$$z = \frac{a-d \pm \sqrt{(d-a)^2+4bc}}{2c} \quad \begin{cases} 1 \text{ if } (d-a)^2+4bc = 0 \\ 2 \text{ if } \neq 0. \end{cases}$$

1.) If φ fixes one point $z_0 \in \hat{\mathbb{C}}$, let ψ be hyp isom sending $z_0 \mapsto \infty$. ~~Show~~ Then $\psi \circ \varphi \circ \psi^{-1}$ is a horiz. trans. Pf: $\varphi(\infty) = \frac{a}{c} = \infty \Rightarrow c = 0$.

$c=0$ and φ has one fixed point so $\sqrt{(d-a)^2+4bc}$ $= \sqrt{(d-a)^2} = d-a = 0 \Rightarrow a = d$. Thus,

$$\frac{az+b}{cz+d} = \frac{az+b}{0+a} = z + b/a. \quad b, a \in \mathbb{C}$$

φ is said to be parabolic (conjugate to horiz. translation in $\mathbb{H}^3$)

2.) If φ fixes two points $z_1, z_2 \in \hat{\mathbb{C}}$. Let ψ be hyperbolic isom sending $z_1 \mapsto 0$, $z_2 \mapsto \infty$.
Then $\varphi(0) = \frac{b}{d} = 0 \Rightarrow b = 0$ and $\varphi(\infty) = \frac{a}{c} \Rightarrow c = 0$

$\Rightarrow \varphi(z) = \dfrac{az+b}{cz+d} = \dfrac{az}{d} = a/d\, z$ (4)

So $\psi \circ \varphi \circ \psi^{-1}$ is a homothety-rotation $z \mapsto a_0 z$ with $a_0 \in \mathbb{C} - \{0\}$. φ fixes the complete hyperbolic geodesic btwn. z_1 and z_2, and acts on the geodesic by translation. If φ fixes every point of g (i.e. $|a_0| = 1$ and $a_0 =$ "$e^{i\theta}$") then φ is called elliptic (pure rotation).
Otherwise φ is called loxodromic (scale + rotate).

Classification of orientation reversing Isoms of $\mathbb{H}^3$:
$\varphi : \mathbb{H}^3 \to \mathbb{H}^3$ is isometric extension of an anti-linear fractional map $\varphi(z) = \dfrac{c\bar{z}+d}{a\bar{z}+b}$ s.t. $ad - bc = 1$

$\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and consider $\varphi^2 = \varphi \circ \varphi$.

$$\varphi^2(z) = \frac{\overbrace{(d\bar{a} + \|c\|^2)}^{a'} z + \overbrace{c\bar{d} + d\bar{b}}^{b'}}{\underbrace{(a\bar{c} + b\bar{a})}_{c'} z + \underbrace{a\bar{d} + \|b\|^2}_{d'}}$$ is a linear fractional trans.

$(a'd' - b'c' = +1)$

φ sends each fixed point of φ^2 to a fixed point of φ^2. If φ^2 has two fixed points and z_1 is one of them then $\varphi^2(z_1) = z_1 \Rightarrow$ if $\varphi(z_1) = z_2$ then $\varphi(\varphi(z_1)) = \varphi(z_2) = z_1$ but then z_2 is also a fixed pt. of φ^2 $\varphi(\varphi(z_2)) = \varphi(z_1) = z_2$

If φ^2 has 1 f.p. and it is z_1 then $\varphi(\varphi(z_1)) = z_1 \Rightarrow \varphi(z_1) = z_1$

otherwise $\varphi(z_1) = z_2$ and z_2 is also a f.p. of φ^2.

1.) Suppose φ^2 is parabolic and $\varphi^2 \neq id$. $\exists$ hyp isom. ψ s.t. $\psi \circ \varphi \circ \psi^{-1}$ is the composition of the euclidean reflection across a vertical euclid. plane w/ a horizontal translation along a vector (nonzero) ~~an~~ parallel to that plane.

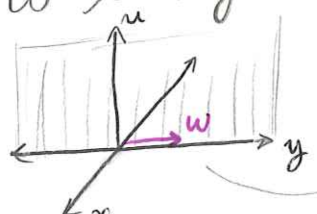
Pf: φ^2 is parab. $\Rightarrow \exists\ \psi$ s.t. $\psi \circ \varphi^2 \circ \psi^{-1} = z + b$ where $b \in \mathbb{C}$ as we proved earlier in our classification of orientation preserving isoms of $\mathbb{H}^3$.

$\Rightarrow (\psi \circ \varphi \circ \psi^{-1})(\psi \circ \varphi \circ \psi^{-1}) = z + b \Rightarrow \psi \circ \varphi \circ \psi^{-1} = -\bar{z} + w$

where $w - \bar{w} = b$ ⊛ (b/c $-(\overline{-\bar{z} + w}) + w = z - \bar{w} + w$)

$\Rightarrow b = y_0 i$ for some y_0. Note that diff. choices of ψ give diff. w satisfying ⊛, but w must always be of the form $x_0 + \frac{1}{2} y_0 i$ so by composing w/ a translation by $-x_0$ we can ensure that $\exists$ an isometry ψ' s.t. $\psi' \circ \varphi \circ (\psi')^{-1}$ is of the form $-\bar{z} + \frac{1}{2} y_0 i$ so w is purely imag. The $-\bar{z}$ is a flip across the y axis of $\mathbb{C} \subset \mathbb{H}^3$ so its extension is a flip across the plane $x = 0$ in $\mathbb{H}^3$ and the vector w being imaginary is parallel to the plane

$x = 0$



$\rightarrow$ note $w = \frac{1}{2} y_0 i$ purely imag. $\Rightarrow$ paralle

2.) Suppose φ^2 fixes two points $z_1, z_2 \in \hat{\mathbb{C}}$ and $\varphi^2 \neq id$. 6

<u>Case 1</u>: Suppose φ also fixes each of z_1, z_2 ($\varphi(z_1) = z_1$, $\varphi(z_2) = z_2$). Then $\exists\ \psi$ s.t. $\psi \circ \varphi \circ \psi^{-1}$ is the composition of a homothety w/ a reflection across a vertical euclidean plane passing through the point 0. $\psi \circ \varphi \circ \psi^{-1}$ fixes 0 and ∞ and is of the form $\frac{c\bar{z}+d}{a\bar{z}+b}$ w/ $ad-bc=1$. 0 fixed $\Rightarrow d=0$ and ∞ fixed $\Rightarrow a=0 \Rightarrow ad-bc=-bc=1$ and $c=-\frac{1}{b}$ so $\psi \circ \varphi \circ \psi^{-1} = \frac{-1}{b^2}\bar{z} = \frac{1}{b^2} \cdot -\bar{z}$. The $-\bar{z}$ reflects across the y-axis of $\mathbb{C} \subset \mathbb{H}^3$ and so $-\bar{z}$ extends to a reflection about the vertical plane, $x=0$. The $1/b^2$ is a homothety and one should note that for any such ψ, $\psi \circ \varphi \circ \psi^{-1} = -\lambda\bar{z}$ where $\|\lambda\| \neq 1$ otherwise, $(\psi \circ \varphi \circ \psi^{-1}) \circ (\psi \circ \varphi \circ \psi^{-1}) = \psi \circ \varphi^2 \circ \psi^{-1} = \lambda\bar{\lambda}z = \|\lambda\|^2 z = z$ but φ^2 fixes <u>exactly</u> two points so $\psi \circ \varphi^2 \circ \psi^{-1} \neq id$. Thus, $\psi \circ \varphi \circ \psi^{-1} = -\lambda\bar{z}$ where λ contributes a homothety-rotation.

φ respects the complete geodesic g from z_1 to z_2 and acts on g by translation.

φ is called orientation reversing loxodromic

3.) Suppose φ^2 fixes z_1 and z_2 and (only) φ exchanges z_1 and z_2 (these are the two cases that can happen since φ fixes the f.p. set of φ^2) Let ψ be isom. sending $z_1 \mapsto 0$ and $z_2 \mapsto \infty$. Then $\psi \circ \varphi^2 \circ \psi^{-1}$ is a scaling/homothety. And $\psi \circ \varphi \circ \psi^{-1}$ exchanges 0 and ∞ $\Rightarrow$ $\frac{c\bar{z}+d}{a\bar{z}+b}$ becomes $\frac{d}{a\bar{z}}$ since $0 \mapsto \infty$ ~~fixed~~

$\Rightarrow \frac{d}{b} = \infty \Rightarrow b=0$ and $\infty \mapsto 0 \Rightarrow \frac{c}{a} = 0 \Rightarrow c=0$

Also though, $\mathcal{Cl}$ is an anti linear fract. trans. $\Rightarrow ad-bc=1$ ⑦

$\Rightarrow ad=1$ in our case and $a=\frac{1}{d}$. Thus,

$\psi\circ\varphi\circ\psi^{-1}=\frac{d^2}{\bar{z}}=d^2\cdot\frac{z}{\|z\|^2}$. Let $\psi'(z)=\frac{1}{d}z$ a hyp.

isometry. Then, $\psi'\circ\psi\circ\varphi\circ\psi^{-1}\circ(\psi')^{-1}(z)=$ ~~...~~

$\psi'\circ(\psi\circ\varphi\circ\psi^{-1})(dz)=\psi'\left(\frac{d^2}{d\bar{z}}\right)=\frac{1}{d}\cdot\frac{d^2}{\bar{d}\bar{z}}=\frac{d^2}{\|d\|^2}\cdot\frac{1}{\bar{z}}=\frac{\lambda}{\bar{z}}$

The $\frac{1}{\bar{z}}=\frac{z}{\|z\|^2}$ contributes an inversion about a circle centered at zero which extends to an inversion about a hemisphere $\subset\mathbb{H}^3$ centered at 0. Also, $\|\lambda\|=\left\|\frac{d^2}{\|d\|^2}\right\|=1$ so $\lambda=\frac{d^2}{\|d\|^2}$ contributes a pure rotation about the u-axis.

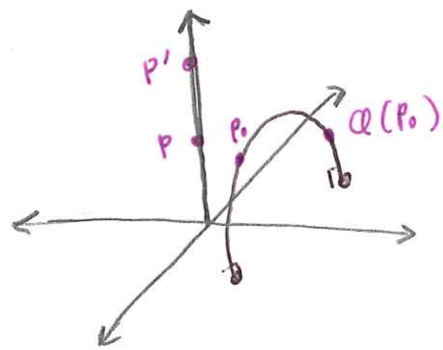
So $\exists\ \tilde{\psi}$ a hyp. isometry s.t. $\tilde{\psi}\circ\varphi\circ\tilde{\psi}^{-1}$ is the composition of an inversion about a hemisphere and a rotation about the u-axis by an angle θ that is not an integer mult. of π. (The restriction on θ comes from the fact that $\tilde{\psi}\circ\varphi^2\circ\tilde{\psi}^{-1}=(\tilde{\psi}\circ\varphi\circ\tilde{\psi}^{-1})\circ(e^{i\theta}/\bar{z})$

$=\frac{e^{i\theta}}{e^{-i\theta}}z=e^{i2\theta}z$ and if $\theta=k\pi$, $2\theta=2\pi k$

$\Rightarrow\tilde{\psi}\circ\varphi^2\circ\tilde{\psi}^{-1}=id$ but this can't be since the isom has exactly $\underline{\underline{2}}$ fixed points.

In this case φ is called orientation reversing elliptic.
(has a f.p. on interior of $\mathbb{H}^3$ namely $(x_0,y_0,u_0)=(0,0,1)$)

4.) Suppose $\varphi^2 = \mathrm{id}$. Choose an arbitrary point $P_0 \in \mathbb{H}^3$ NOT fixed by φ. (Such a point exists since $\varphi \neq \mathrm{id}$). Let g be the unique complete geodesic btwn. $\varphi(P_0)$ and P_0, and let Ψ be the hyperbolic isom. taking g to the geodesic btwn 0 and ∞ in $\mathbb{H}^3$.



Then $\Psi \circ \varphi \circ \Psi^{-1}$ is an isom. of $\mathbb{H}^3$ taking a point $P \in [0, \infty]$ to $P' \in [0, \infty]$ as in the figure above. The possible isoms. that do this are:

$(0,0,u) \longmapsto (0,0,\lambda u) \quad \lambda \in \mathbb{C} - \{0\}$

$(0,0,u) \longmapsto (0,0,1/u)$ inversion about a sphere

(trans. won't do it cause $(0,0,u)$ would map to $(x_0, y_0, u) \notin [0,\infty]$).

~~So~~ $\Psi \circ \varphi \circ \Psi^{-1}$ must be an antilinear frac. trans. as well. Thus, $\Psi \circ \varphi \circ \Psi^{-1}$ is either an inversion about a sphere (ie. $\Psi \circ \varphi \circ \Psi^{-1} = \frac{1}{\bar{z}} = \frac{z}{\|z\|^2}$) or $\Psi \circ \varphi \circ \Psi^{-1}$ is

(many fixed points in $\mathbb{H}^3$)

the composition of such an inversion with a homothety (ie. $\Psi \circ \varphi \circ \Psi^{-1} = \frac{\lambda}{\bar{z}}$) ~~[illegible]~~

~~[illegible]~~ But for the homothety we can simplify. First note that $\Psi \circ \varphi^2 \circ \Psi^{-1} = \frac{\lambda}{\bar{\lambda}} z = z \Rightarrow \lambda/\bar{\lambda} = 1 \Rightarrow \lambda = \bar{\lambda}$

and thus $\lambda \in \mathbb{R} - \{0\}$. If $\lambda > 0$ then let $\psi'(z) = \frac{1}{\sqrt{\lambda}} z$. Then $\exists\, \tilde{\psi}$ s.t. $\tilde{\psi} \circ \varphi \circ \tilde{\psi}^{-1} = \psi' \circ \psi \circ \varphi \circ \psi^{-1} \circ (\psi')^{-1} = \frac{1}{\sqrt{\lambda}} \cdot \frac{1}{\overline{\sqrt{\lambda}}} \cdot \frac{\lambda}{\bar{z}} = \frac{\lambda}{\sqrt{\lambda}\sqrt{\lambda}} \cdot \frac{1}{\bar{z}}$

$= \frac{1}{\bar{z}}$ (b/c $\lambda \in \mathbb{R}_{>0}$ $\overline{\sqrt{\lambda}} = \sqrt{\lambda}$) thus we are in the case of a pure inversion about a sphere. For $\lambda < 0$ let $\psi' = \frac{1}{\sqrt{\lambda}} z$.

Then $\exists\, \tilde{\psi}$ s.t. $\tilde{\psi} \circ \varphi \circ \tilde{\psi}^{-1} = \frac{1}{\sqrt{\lambda}} \cdot \frac{1}{\overline{\sqrt{\lambda}}} \cdot \frac{\lambda}{\bar{z}} = \frac{\lambda}{-\lambda} \cdot \frac{1}{\bar{z}}$

(if $\sqrt{\lambda} = ri$ then $\sqrt{\lambda} \cdot \overline{\sqrt{\lambda}} = r^2$ and $\lambda = -r^2$).

So $\tilde{\psi} \circ \varphi \circ \tilde{\psi}^{-1} = \frac{-1}{\bar{z}} = \frac{e^{i\pi}}{\bar{z}} \Rightarrow$ inversion about sphere composed with a rotation by $\pi = \theta$ about u-axis.

So summarizing the above gives the following list of ALL isometries of $\mathbb{H}^3$:

<u>Orientation Preserving</u>: $\varphi(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$, $a, b, c, d \in \mathbb{C}$ $\Big\}$ isom. of $\mathbb{H}^3$ is an ext. of this isom on $\hat{\mathbb{C}}$.

1.) Parabolic: φ fixes exactly one point in $\hat{\mathbb{C}} \subseteq \mathbb{H}^3$.
φ is conjugate to an isom. of the form $z \mapsto z + a$ where $a \in \mathbb{C} - \{0\}$
The extension being an isom of $\mathbb{H}^3$ corresponding to a horiz. trans.

2.) a.) homothety: φ fixes exactly two points in $\hat{\mathbb{C}}$
φ is conjugate to an isom. of the form $z \mapsto \lambda z$ where $\lambda \in \mathbb{R}_{>0}$
So the extension is an isometry of $\mathbb{H}^3$ corr. to a pure scaling by a real number.

b.) Loxodromic: φ fixes exactly two points in $\hat{\mathbb{C}}$
φ is conjugate to an isom of the form $z \mapsto \lambda z$ where $\lambda \in \mathbb{C} - \{0\}$
The extension is an isom. of $\mathbb{H}^3$ corr. to scaling and rotating about u-axis

c) Elliptic (Pure Rotation): φ fixes every point of a geod. g in $\mathbb{H}^3$. φ is conj. to an isom. of the form $z \mapsto \lambda z$ where $\lambda \in \mathbb{C} - \{0\}$ and $|\lambda| = 1$ $\Rightarrow$ of the form $z \mapsto e^{i\theta} z$. The extension is an isom. of $\mathbb{H}^3$ that is a pure rotation about the u-axis by an angle θ.

Orientation Reversing Isoms: $\varphi(z) = \dfrac{c\bar{z}+d}{a\bar{z}+b}$ $\quad a,b,c,d \in \mathbb{C}$, $ad-bc=1$.
isoms are extensions of the isoms on $\hat{\mathbb{C}}$ to all of $\mathbb{H}^3$.

1.) orientation reversing parabolic: φ has one fixed point in $\hat{\mathbb{C}}$. φ is conjugate to an isom. of the form $z \mapsto -\bar{z} + ri$ and the extension to $\mathbb{H}^3$ is the composition of a Euclidean reflection across a vertical plane (the uy-axis) with a horiz. translation along a vector (nonzero) parallel to that plane.

2.) Orientation reversing loxodromic: φ has two fixed points in $\hat{\mathbb{C}}$. φ is conj. to an isom. of the form $z \mapsto -\lambda\bar{z}$ where $|\lambda| \neq 1$ so the extension to $\mathbb{H}^3$ is the composition of a homothety-rotation with a reflection across a vertical euclidean plane passing thru the point O (uy-axis etc.).

3.) Orientation Reversing Elliptic: φ has a fixed point in the interior of $\mathbb{H}^3$. φ is conj. to an isom. of the form $z \mapsto \dfrac{\lambda}{\bar{z}}$ where $|\lambda| = 1$, so the extension is an isom. of $\mathbb{H}^3$ that is a composition of ~~an~~ an inversion about a hemisphere $\subset \mathbb{H}^3$ centered at O and a pure rotation about the u-axis by an angle θ that is not an integer mult. of π.

4.) φ is an isom. of $\mathbb{H}^3$ that is a pure inversion about a sphere ($z \mapsto \frac{1}{\bar{z}}$) or $z \mapsto \dfrac{-1}{\bar{z}} = \dfrac{e^{i\pi}}{\bar{z}}$ an inversion about sphere plus rotation by π. (1 fixed pt. in $\mathbb{H}^3$)
↳ Fixed pts. on hemisp.

Now that we know all of the isoms. of $\mathbb{H}^3$ we can show which of these can be elements of deck trans. groups for hyperbolic 3-mfds.

The only options are OP 2a, OP2b, and OR 2.

The parabolic isoms. are ruled out due to the same reasoning as in the 2D case.

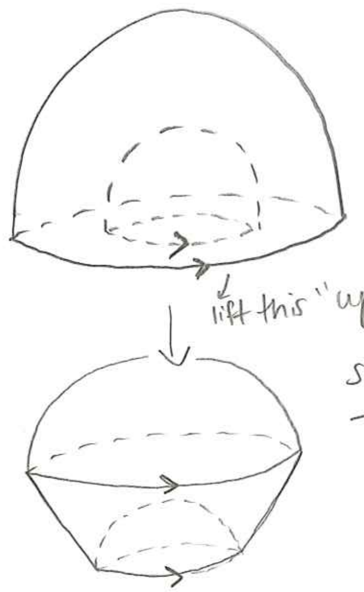
Next we will study what the $\mathbb{Z}$-covers of a hyperbolic 3-mfd. could be.

<u>Case 1</u> : (OP2a) orientation preserving homothety.

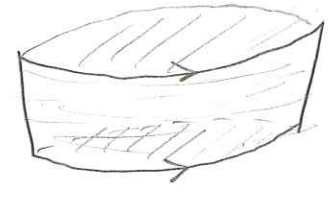
Suppose $\varphi(z)$ is the extension of the isom. $z \mapsto Rz$ where $R \in \mathbb{R}_{>0}$. A fund. region for the action of φ on $\mathbb{H}^3$ is the region between the hemisphere of radius 1 and the hemisphere of radius R.

the regions in $\hat{\mathbb{C}}$ are not included / open.

identifying points in H_1 to their images in H_R:



shrink/collapse → hemispheres

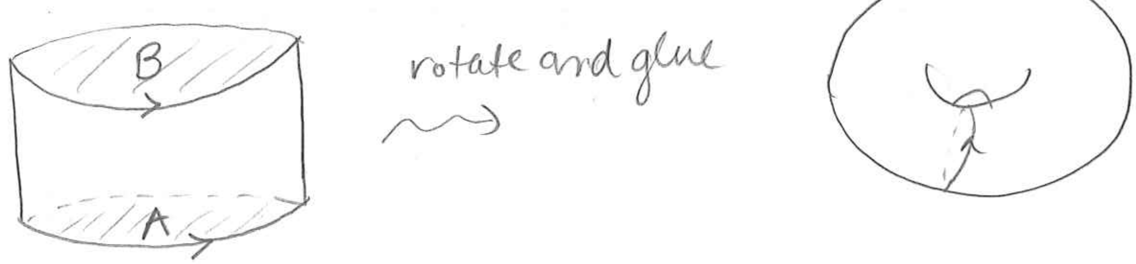


(solid)

Identify two discs which were the two hemispheres

open, solid torus

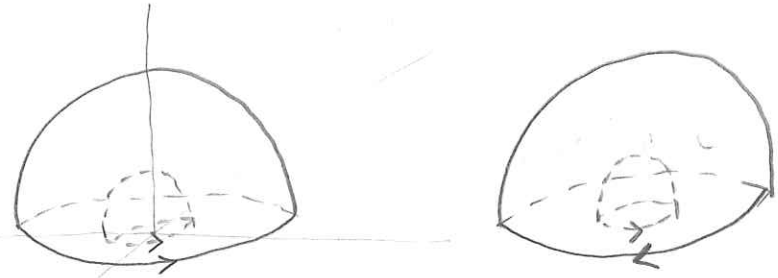
Case 2 (OP2b): This is orientation preserving loxodromic case, where the only difference from case 1 is the additional rotation by θ. However, this rotation can simply be interpreted as a rotation of face (aka disc) A by θ before gluing to face / disc B: (12)

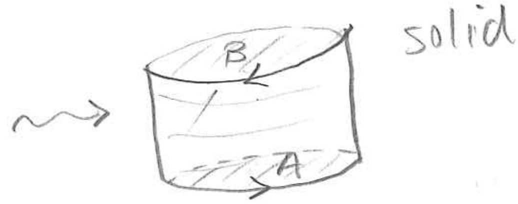


Topologically we still have an open, solid torus (just with a twist)

Case 3: (OR2) orientation reversing loxodromic (or homothety) The difference from case 1 and 2 lies in the fact that H_1 corresponding to disc/face A is glued to H_R (corr. to disc/face B) ~~and~~ after a reflection across the xy-axis (or some other vertical plane thru O).

Instead of: we have:





this identification results in