Hyperbolic Geometry:

The hyperbolic metric on $\mathbb{H}^2$ is $ds = \frac{\sqrt{dx^2+dy^2}}{y}$.

Notice this is the sq. root of the riemannian metric we usually use. That's because we are usually looking at vectors in the tangent space of $\mathbb{H}^2$ and the "dot product" there is indeed given by $\frac{dx^2+dy^2}{y^2}$.

Let $I = [0,1]$ and $\gamma : I \to \mathbb{H}^2$ be a p. wise diff. path $\gamma = \{z(t) = x(t) + iy(t) \in \mathbb{H}^2 : t \in I\}$ then hyperbolic length $\ell(\gamma)$ is given by:

$$\int_\gamma \frac{\sqrt{dx^2+dy^2}}{y} = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)}\, dt = \int_0^1 \frac{\left|\frac{dz}{dt}\right|}{y(t)}\, dt$$

Geodesics in $\mathbb{H}^2$ are lines and semi circles orthogonal to $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$. To see this let's first talk about the isom. of $\mathbb{H}^2$: Consider the Möbius transformations:

$\left\{ z \mapsto \frac{az+b}{cz+d} : ad-bc = 1 \right\}$ we can represent these by matrices in $SL(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad-bc = 1 \right\}$

but notice that for the Möb trans. $\frac{az+b}{cz+d} = \frac{-az-b}{-cz-d}$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ "=" $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ so our transformations are actually represented by $PSL(2,\mathbb{R}) = SL(2,\mathbb{R}) / \{\pm I\}$.

These are the orientation preserving isometries of $\mathbb{H}^2$.

Notice that even if $\dfrac{az+b}{cz+d}$ is s.t. $ad-bc \neq 1$ but $ad-bc > 0$

we can call $ad-bc = \Delta$ and multiply $\dfrac{az+b}{cz+d}$ by $\dfrac{\sqrt{\Delta}}{\sqrt{\Delta}}$

$\Rightarrow \dfrac{az+b}{cz+d} \cdot \dfrac{\sqrt{\Delta}}{\sqrt{\Delta}}$ has det 1 !

So $PSL(2,\mathbb{R})$ contains all trans. $\dfrac{az+b}{cz+d}$ s.t. $\Delta > 0$.

Note we have shown these are indeed isom-s of $\mathbb{H}^2$ in problems on Riemannian Geo. by showing $\varphi^* g = g$. where $g = \dfrac{dx^2+dy^2}{y^2}$.

Now take two points in $\mathbb{H}^2$, $z_1 = ia$, $z_2 = ib$ where $b > a$.

If $\gamma : I \to \mathbb{H}$ is a path from z_1 to z_2 then

$$\ell(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)}\, dt \geq \int_0^1 \frac{|dy/dt|}{y(t)}\, dt \geq \int_0^1 \frac{\frac{dy}{dt}}{y}\, dt$$

with eq. iff $\frac{dx}{dt} = 0$ No variance in x dir.

path goes "up" $\Rightarrow$ eq.

$$= \int_a^b \frac{dy}{y} = \ln(b) - \ln(a) = \ln(b/a).$$

Thus the curve between z_1 and z_2 that has smallest length will have $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} > 0$.

take $\tilde{\gamma}(t) = 0 + i\left(a(1-t) + bt\right) = x(t) + i\, y(t)$.

$\Rightarrow x'(t) = 0 \quad y'(t) = b - a > 0$.

Then $\ell(\tilde{\gamma}) = \int_0^1 \frac{dy}{dt}/y \, dt = \int_a^b \frac{dy}{y} = \ln(b/a)$.

So the geodesic between any two points on Im axis will be the ~~unique~~ path(!) $\tilde{\gamma}$ that is the portion of the Im axis between ai and bi.

Now we will deal with the arbitrary case: ②

Let L be the unique circle or straight line orthogonal to $\mathbb{R}$ passing thru $z_1, z_2 \in \mathbb{H}^2$.

We want to show that for a suitable β, $T(z) = \frac{-1}{z-\alpha} + \beta$, where α is the/a point in $\mathbb{R} \cap L$, takes L to Im axis.

Now $T(z) = \dfrac{-1 + \beta z - \beta\alpha}{z-\alpha} = \dfrac{\beta z + (-1 - \alpha\beta)}{z + (-\alpha)}$ is in $PSL(2,\mathbb{R})$:

$$\det \begin{bmatrix} \beta & (-1-\alpha\beta) \\ 1 & -\alpha \end{bmatrix} = -\beta\alpha + 1 + \alpha\beta = 1.$$

So T is an isometry and thus preserves the length of L and the distance between z_1 and z_2.

Now suppose L is straight line and let $\beta = 0$.

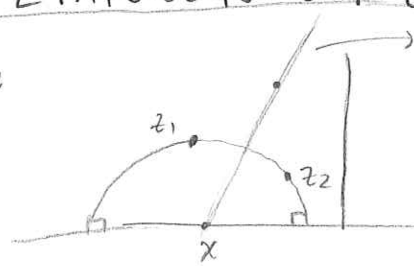
Then $T(\alpha) = \dfrac{-1}{\alpha - \alpha} = \infty \qquad T(\infty) = 0$

and since T is a conformal map, $T(L)$ must intersect $\mathbb{R} \cup \{\infty\}$ at an angle of $\pi/2$. $\Rightarrow T(L) = \text{Im axis}$ (contains 0 and ∞)

(If you want to see another way, $T(z_1) = \frac{-1}{iy_1} \in \text{Im Axis}$ & $T(z_2) = \frac{-1}{iy_2}$)

Now if L is the unique semi-circle thru z_1 and z_2 and intersecting $\partial\mathbb{H}^2$ at $90°$ then L intersects $\partial\mathbb{H}^2$ at α and α'.

L is indeed unique:



this line represents all pts. equidistant from z_1 and z_2. This int. $\mathbb{R}$ at a point, x, that is the unique point on $\mathbb{R} \cap$ the line and thus $\exists$ a unique semi circle w/ center x.

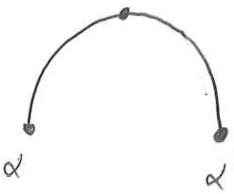
Now since $\alpha \mapsto \infty$ under $T(z)$ we need $\alpha' \mapsto 0$. So

$T(\alpha') = \dfrac{-1}{\alpha' - \alpha} + \beta = 0 \Rightarrow \beta = \dfrac{1}{\alpha' - \alpha}$. Again $T(z)$ Conformal.

$\Rightarrow T(L) = \text{Im Axis}$

To verify for yourself check the point $z_0 = \frac{\alpha'+\alpha}{2} + \frac{\alpha'-\alpha}{2} i$ on L

$z_0 = \frac{\alpha'+\alpha}{2} + i\frac{\alpha'-\alpha}{2}$



$\alpha \qquad \alpha'$

$T(z_0) \in \text{Im Axis}.$

Thus $T(z) = \frac{-1}{z-\alpha} + \frac{1}{\alpha'-\alpha}$ brings L to Im axis.

To finish the proof recall T is an isom. We know that for two points on the Im axis the shortest path from one to the other, the geodesic between them, is the part of the Im axis between them. We have just showed that for two arb. points in $\mathbb{H}^2$ the segment of the straight line or semi-circle thru them is taken to the portion of the Im axis between them when they are transformed onto the Im axis via an isom. $\Rightarrow$ The geodesic between these points is the portion of L between them (isoms take geos to geos!!). ∎

Thm: Any trans. in $PSL(2,\mathbb{R})$ maps geodesics to geodesics.
Prove using the fact that $\rho(z,w) = \rho(z,\xi) + \rho(\xi,w)$ iff $\xi \in [z,w] \rightarrow$ the geod. conn. z & w.

Cross Ratio: The cross ratio of four distinct points in $\hat{\mathbb{C}}$ is

$$(z_1, z_2; z_3, z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

If $z, w \in \mathbb{H}$ $z \neq w$ let z^*, w^* be the endpoints of the geo. conn. z, w where z is between z^* and w. Then

$d(z,w) = \ln(w, z^*; z, w^*)$

Note: Feng's method gives us (w, z^*, w^*, z) instead.

unit disc model: $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ $ds = \frac{2\sqrt{dz d\bar{z}}}{1-|z|^2} = \frac{2|dz|}{1-|z|^2}.$ (3)

ALL Isometries of $\mathbb{H}^2$:

We aim to show that all isometries of $\mathbb{H}^2$ are:

(1) $\varphi \in PSL(2,\mathbb{R})$ $\varphi(z) = \frac{az+b}{cz+d}$ $ad-bc=1$

(2) $\varphi(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ $ad-bc= -1$

Outline: We know trans. in $PSL(2,\mathbb{R})$ take geos $\mapsto$ geos and that $PSL(2,\mathbb{R}) \subset \text{Isom}\,\mathbb{H}$.

Now let φ be some isometry of $\mathbb{H}^2$. $\exists\, g \in PSL(2,\mathbb{R})$ s.t. $g \circ \varphi(I) = I$ where I = im axis. and $g \circ \varphi \in \text{Iso}(\mathbb{H}^2)$

Apply some $z \mapsto kz$ and $z \mapsto -\frac{1}{z}$ as necc. to make sure $g \circ \varphi(i) = i$.

$\Rightarrow$ every other point of I is fixed because $\rho(i, bi) = \ln(b)$ $= \rho(i, g\circ\varphi(bi))$ $\Rightarrow$ $g \circ \varphi(bi) = bi$ (Note: suppose we consider $1/2\, i$ then $g\circ\varphi(1/2\, i)$ could be $3/2\, i$ since $\rho(i, 3/2\, i) = \rho(i, 1/2\, i)$ but in that case we could alter g slightly by composing with $-\frac{1}{z}$ to make sure we don't invert about i.) Thus, $g\circ\varphi$ fixes every point on I.

Now if we were only concerned with the orientation preserving isom. of $\mathbb{H}^2$ then $g\circ\varphi$ is analytic but $g\circ\varphi(z) - z = 0$ for infinitely many values $\Rightarrow$ $g\circ\varphi(z) \equiv z$ $\Rightarrow$ $\varphi(z) = g^{-1}(z) \in PSL(2,\mathbb{R})$.

For the general case we need to do a little more work $\longrightarrow$

$z = x+iy \in \mathbb{H}$, $g \circ \varphi(z) = u+iv$

$\rho(z, it) = \rho(g\circ\varphi(z), g\circ\varphi(it)) = \rho(u+iv, it)$

claim: ~~Now~~ $\sinh(x) = \frac{1}{2}(e^{x} - e^{-x}) \Rightarrow \sinh\left(\frac{1}{2}\rho(z,w)\right) = \frac{|z-w|}{2(\mathrm{Im}(z)\mathrm{Im}(w))^{1/2}}$

invariant under $PSL(2,\mathbb{R})$

so $\rho(ia, ib) = \ln(b/a)$

$$\sinh\left(\frac{1}{2}\ln(b/a)\right) = \frac{1}{2}\frac{(b-a)}{(ab)^{1/2}} = \frac{|z-w|}{2(\mathrm{Im}(z)\mathrm{Im}(w))^{1/2}}$$

thus for any $z, w \in \mathbb{H}$ this holds.

thus

~~and~~ $\rho(z, it) = \rho(u+iv, it) \Rightarrow \frac{|z-it|}{2(\mathrm{Im}(z)\,t)^{1/2}} = \frac{|u+iv-it|}{2(vt)^{1/2}}$

$$= \frac{|x+i(y-t)|}{2(yt)^{1/2}} = \frac{|u+i(v-t)|}{2(vt)^{1/2}} \xRightarrow{\text{with some work}}$$

$v = y$ and thus $x^2 = u^2 \Rightarrow x = u$ or $-x = u$.

$\Rightarrow g\circ\varphi(z) = z$ or $g\circ\varphi(z) = -\bar{z}$

$\Rightarrow \varphi \in PSL(2,\mathbb{R})$ or $\varphi(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ s.t. $ad-bc = -1$.

Hyperbolic Area is invariant under $PSL(2,\mathbb{R})$: $\det(DT) =$

$T(z) = \frac{az+b}{cz+d} = u+iv = w$ $\quad \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$

$= \left|\frac{dT}{dz}\right|^2 = \frac{1}{|cz+d|^4}$ $\quad T'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{1}{(cz+d)^2}$

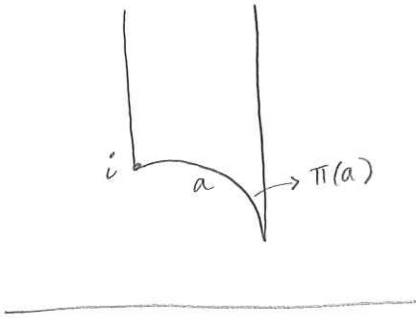
$\mu(T(A)) = \int_{T(A)} \frac{du\,dv}{v^2} = \int_A \frac{\frac{1}{|cz+d|^4}}{v^2}\,dx\,dy$ $\quad$ Now $\frac{1}{v^2} = \frac{1}{\mathrm{Im}\left(\frac{az+b}{cz+d}\right)^2}$

$= \frac{|cz+d|^4}{y^2}$

$= \int_A \frac{1}{|cz+d|^4} \cdot \frac{|cz+d|^4}{y^2}\,dx\,dy = \int_A \frac{dx\,dy}{y^2} = \mu(A)$. $\blacksquare$

Angle of Parallelism :

Take a hyperbolic triangle with angles $0, \pi/2$, and $\alpha \neq 0$. Denote the only finite side of the triangle a. The angle of parallelism α is a function of a : $\alpha = \Pi(a)$.



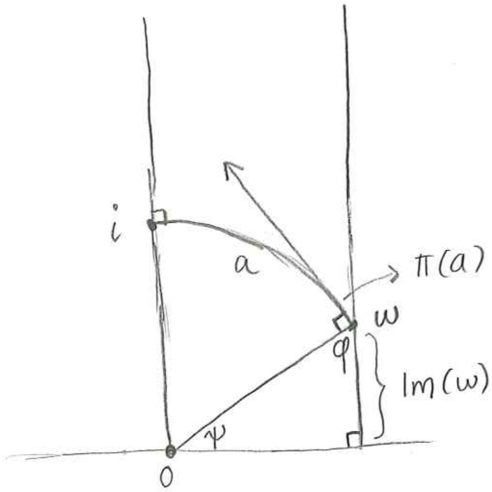
The following relations hold :

(i) $\tan(\Pi(a)) = \dfrac{1}{\sinh(a)}$

(ii) $\sin(\Pi(a)) = \dfrac{1}{\cosh(a)}$

(iii) $\sec(\Pi(a)) = \dfrac{1}{\tanh(a)}$

We will show (ii) and show (i) and (iii) are equivalent :



Since the angle at i is $\pi/2$ we know the center of the circle with segment a must be zero (only place where tang. vector is $\pi/2$)

$$\Pi(a) + \frac{\pi}{2} + \varphi = \pi$$

Also though $\psi + \frac{\pi}{2} + \varphi = \pi$

$\Rightarrow \Pi(a) = \psi$. Also $|w| = 1$ since radius of circle is $|i| = 1$. So the hypotenuse of Δ with angles $\psi, \varphi, \pi/2$ is length 1.

Thus $\sin(\Pi(a)) = \sin(\psi) = \dfrac{\operatorname{Im}(w)}{1} = \operatorname{Im}(w)$.

Now we have the formula $\cosh(a) = 1 + \dfrac{|i-w|^2}{2\operatorname{Im}(i)\operatorname{Im}(w)}$

$= 1 + \dfrac{|i-w|^2}{2\sin\Pi(a)}$ Now $|i-w|^2 = |i - \sin\Pi(a)\, i - \cos\Pi(a)|^2$

$= (1-\sin\Pi(a))^2 + \cos^2\Pi(a) = 2 - 2\sin\Pi(a)$

$$\Rightarrow \cosh(a) = 1 + \frac{2-2\sin\Pi(a)}{2\sin\Pi(a)} = \frac{\sin\Pi(a) + 1 - \sin\Pi(a)}{\sin\Pi(a)} = \frac{1}{\sin\Pi(a)}$$

$$\Rightarrow \sin \Pi(a) = \frac{1}{\cosh(a)}$$

To prove i and iii:

$$\cos \Pi(a) = \sqrt{1-\sin^2 \Pi(a)} = \sqrt{1-\frac{1}{\cosh^2(a)}} = \sqrt{\frac{\cosh^2(a)-1}{\cosh^2(a)}}$$

$\underline{\cosh^2(a)-\sinh^2(a)=1}$

$$\sqrt{\frac{\sinh^2(a)}{\cosh^2(a)}} = \tanh(a) \Rightarrow \frac{1}{\cos \Pi(a)} = \sec \Pi(a) = \frac{1}{\tanh(a)}$$

proving iii

$$\text{Now } \frac{\sin \Pi(a)}{\cos \Pi(a)} = \frac{\frac{1}{\cosh(a)}}{\tanh(a)} = \frac{1}{\sinh(a)}$$

proving i.

Hyperbolic Trig Identities:

Consider hyperbolic triangle with sides of hyperbolic length a, b, c (all finite) and opposite angles α, β, γ:

Thm: The sine rule — $\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh(c)}{\sin \gamma}$

Cosine Rule I — $\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos\gamma$

Proved in hyp. geo. problems.

Cosine Rule II — $\cosh(c) = \frac{\cos\alpha \cos\beta + \cos\gamma}{\sin\alpha \sin\beta}$

The group $PSL(2, \mathbb{R}) = \left\{ z \mapsto T(z) = \frac{az+b}{cz+d} \;\middle|\; ad-bc=1 \right\}$:

We claim that there are 3 types of orientation preserving isos. of $\mathbb{H}^2$, that is we can classify $PSL(2, \mathbb{R})$.

1. Parabolic isometry: 1 fixed point on $\partial \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ ex: $\varphi(z) = z + \alpha$, $\alpha \in \mathbb{R}$
2. Hyperbolic: 2 fixed points on $\partial \mathbb{H}^2$. ex: $\varphi(z) = \lambda z$ $\lambda \in \mathbb{R}$.
3. elliptic: 1 fixed point in $\mathbb{H}^2$ ex: rotation about i in $\mathbb{H}^2$.

What do the matrix rep. in $PSL(2,\mathbb{R})$ for these examples look like?:

$\varphi(z) = z+\alpha$ $\quad A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \in PSL(2,\mathbb{R})$

$\varphi(z) = \lambda z$ $\quad A' = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$ but $\dfrac{\lambda z}{1} \cdot \dfrac{\frac{1}{\sqrt{\lambda}}}{\frac{1}{\sqrt{\lambda}}} = \dfrac{\sqrt{\lambda}\, z}{\frac{1}{\sqrt{\lambda}}}$ $\quad A = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix}$

$\varphi(z)$ is a rotation about i: $\dfrac{ai+b}{ci+d} = i$ $\quad ai+b = di-c$

$\varphi(z) = \dfrac{az+b}{cz+d}$ $\quad \Rightarrow a=d, \ -b=c$

$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ normalize by $\frac{1}{\sqrt{\det}}$ if needed so that $a^2+b^2=1$.

$\Rightarrow A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

We claim that if A is the matrix $\in PSL(2,\mathbb{R})$ of φ then

$(tr(A))^2 = 4 \iff \varphi$ is parabolic

$(tr(A))^2 > 4 \iff \varphi$ is hyperbolic

$tr(A)^2 < 4 \iff \varphi$ is elliptic

See hyp. geo. problems.

Now we move onto the discussion of hyperbolic polygons, convexity Fuchsian groups and Fundamental domains (using Beardon).

We call the circle of points at ∞ $= \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}$.

geod. btwn z and w

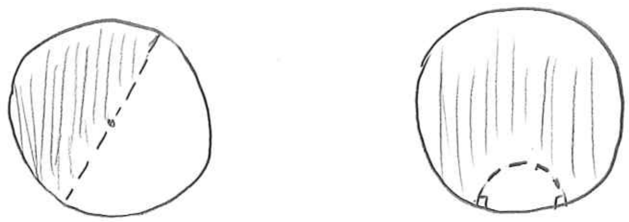
A set in $\mathbb{H}$ is convex iff $\forall\, z,w$ in the set, $[z,w]$ is in the set

If $\mathcal{E} \subset \mathbb{H}$ convex then so are $\mathcal{E}^\circ$ and $\overline{\mathcal{E}}$. If $\mathcal{E}_1, \mathcal{E}_2, \ldots$ are convex and $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3 \subset \cdots$ then $\bigcup \mathcal{E}_n$ is convex.

If $\mathcal{E}_\alpha$ are convex for all α then $\bigcap_\alpha \mathcal{E}_\alpha$ is convex.

Def : A half plane in $\mathbb{H}$ is one component of the compliment of a geodesic in $\mathbb{H}$.

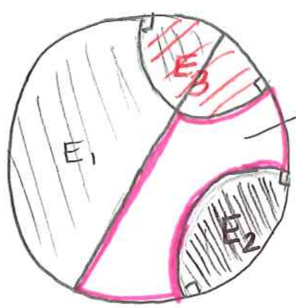
Ex :



Half planes and ~~gt~~ closed half planes (incl. geod. segment) are convex in $\mathbb{H}$.

If E_α is a family of half planes then $(\bigcup E_\alpha)^c = \bigcap E_\alpha^c$ is the intersection of convex sets $\Rightarrow$ its convex.

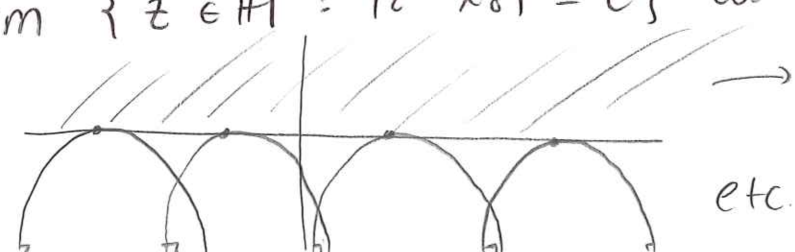
Ex :



$\left(\bigcup_{i=1}^{3} E_i\right)^c = \bigcap_{i=1}^{3} E_i^c$ is convex.

Def : A horocycle region is the interior of a Euclid. circle which is tangent to the @ ∞. A horocycle is the boundary of a horo region

In $\mathbb{H}$ with ∞ as the point of tangency we can take a horocycle region to be $\{x+iy : y>t\}$. This is convex as the compliment of the union of all half planes of the form $\{z \in \mathbb{H} : |z-x_0| \leq t\}$ as x_0 varies over $\mathbb{R}$:



A set is locally convex iff $\forall z \in E$ $\exists N$ an open nbd. around z s.t. $N \cap E$ is convex. 6

Theorem: A closed subset E of the hyperbolic plane (or the eucl. plane) is convex iff its conn. and locally convex.

Polygons:

A polygon is the interior of a closed Jordan curve

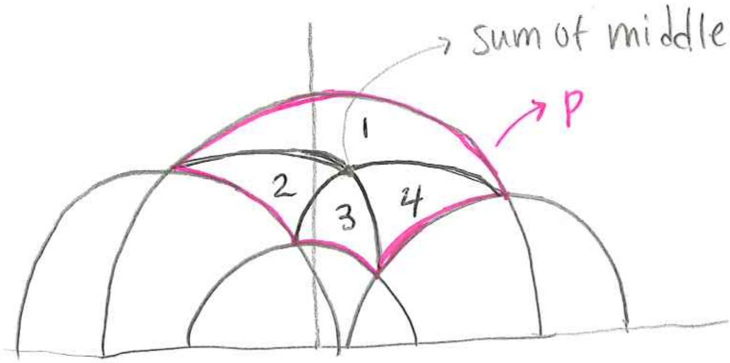
$[z_1, z_2] \cup \cdots \cup [z_n, z_1]$ these are geodesic segments ("lines")

θ_j is the interior angle at z_j (the vertices could be in $\mathbb{R} \cup \{\infty\}$)

Theorem: if P is a polygon w/ interior angles $\theta_1, \dots, \theta_n$ then $\text{Area}(P) = (n-2)\pi - (\theta_1 + \cdots + \theta_n)$

Subdivide into triangles:

sum of middle angles is 2π



n-gon break into n triangles where each has area $\pi - \alpha - \beta - \gamma$ so we get

$$n\pi - \theta_1 - \cdots - \theta_n - \underbrace{2\pi}_{\text{middle angles}}$$

Convex Polygons:

Theorem: Let P be a polygon w/ angles $\theta_1, \dots, \theta_n$ then P is convex iff $0 \le \theta_j \le \pi$ $\forall j \in [1, n]$.

"$\Rightarrow$"

This follows immediately from the fact that P is convex iff P conn. + locally convex for suppose P is convex and $\theta_j > \pi$. then we cannot have local convexity at z_j:

in any nbd. of z_j $\exists z, w$ two points in that nbd. s.t. $[z, w]$ not in that nbd.

"$\Leftarrow$" This is harder and requires and makes use of the Klein model for hyperbolic space, which we will discuss later.

<u>Theorem</u>: Let $\theta_1, \dots, \theta_n$ be any ordered n-tuple with $0 \le \theta_j < \pi$ $j = 1, \dots, n$. Then there exists a polygon P w/ int. angles $\theta_1, \dots, \theta_n$ occuring in this order around ∂P iff

$\theta_1 + \cdots + \theta_n < (n-2)\pi$ (Proof is a construction that is not hard P. 155 Beardon (the geom. of discrete grps)

<u>Application of Thm</u>: $\exists$ a polygon w/ all $\theta_j = \pi/2 \iff n \ge 5$

<u>Fuchsian Groups</u>:

<u>Def</u>: Take A topological space X and a group G that acts on X by homeomorphisms. We say G acts discontinuously on X iff $\forall$ cpt. $K \subset X$, $g(K) \cap K = \emptyset$ except for a finite number of elements g in G.

Suppose that G acts discont. on X then:

1.) Every subgroup of G acts discont. on X.

2.) If φ is a homeo of X onto Y then $\varphi G \varphi^{-1}$ acts discontinuously on Y.

3.) If Y is a G-invarient subset of X, then G acts discontin. on Y

4.) If $x \in X$ and $g_1, g_2, \dots$ are distinct then $g(x_1), g(x_2), \dots$ cannot converge to any y in X.

5.) If $x \in X$ then the stabilizer G_x is finite

Def: A group G of Mobius transformations is a Fuchsian group iff there is some G-invariant disc in which G acts discontinuously.

Def: A subset D of the hyperbolic plane is a fundamental domain for a Fuchsian group G iff

(1) D is a domain
(2) there is some fundamental set F w/ $D \subset F \subset \tilde{D}$
(3) h-area $(\partial D) = 0$

Here a fundamental set for G is a subset F of $\mathbb{D}$ (or $\mathbb{H}$) which contains exactly one point from each orbit in $\mathbb{D}$. Thus no two points in F are G-equivalent and

$$\bigcup_{g \in G} g(F) = \mathbb{D}.$$

if D is a fundamental domain, then for all g in G

$(g \neq I)$ $\quad g(D) \cap D = \emptyset \qquad \bigcup_{g \in G} g(\tilde{D}) = \Delta = \mathbb{D}$

and we say that D and its images tesselate $\Delta = \mathbb{D}$

For a hyperbolic surface Σ we can think of a fund. domain as any map of a simply conn. space (such as a disc D^2) into Σ because this map will lift to univ. cover $\mathbb{H}$ ($\mathbb{D}$) and will tesselate $\mathbb{H}$ (or $\mathbb{D}$) by its image under deck trans. grp.

Classification of orientation Reversing Isometries of $\mathbb{H}^2$:

$$f(z) = \frac{a(-\bar{z}) + b}{c(-\bar{z}) + d} \qquad \text{where } ad - bc = 1$$

Classify by fixed points:

$$\frac{-a\bar{z} + b}{-c\bar{z} + d} = z \iff -a\bar{z} + b = -c|z|^2 + dz$$

$$\iff -a\bar{z} + b + c|z|^2 - dz = 0 = g(z) = u(x,y) + i\,v(x,y)$$

$$g(z) = 0 \iff u(x,y) = 0 \text{ and } v(x,y) = 0$$

$$u(x,y) = -ax + b + cx^2 + cy^2 - dx$$
$$v(x,y) = ay - dy$$

So $v(x,y) = (a-d)y = 0 \iff a = d$ or $y = 0$

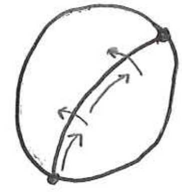
If $y = 0 \Rightarrow u(x,y) = cx^2 + (-a-d)x + b = 0$

when $x = \dfrac{a+d \pm \sqrt{(a+d)^2 - 4bc}}{2c} = \dfrac{a+d \pm \sqrt{a^2 + d^2 + 2ad - 2bc - 2bc}}{2c}$

$$= \frac{a+d \pm \sqrt{a^2 + d^2 + 2(ad - bc) - 2bc}}{2c} = \frac{a+d \pm \sqrt{a^2 + d^2 + 2 - 2(ad-1)}}{2c}$$

$$= \frac{a+d \pm \sqrt{(a-d)^2 + 4}}{2c} \Rightarrow \text{two real roots}$$

$x = \;$) $\qquad y = 0$



$\Rightarrow f(z)$ is orient. reversing hyperbolic isom.

<u>Note</u>: In this case we still have a fixed geodesic axis. ②

If $a = d$ then $a^2 - bc = 1$ ⊛ and

$$u(x,y) = cx^2 - 2ax + b + cy^2 = 0$$

① if $c \neq 0$:

$$c\left(x^2 - \frac{2a}{c}x + y^2\right) = -b$$

$$c\left(x^2 - \frac{2a}{c}x + \frac{a^2}{c^2} + y^2\right) = -b + \frac{a^2}{c} \overset{⊛}{=} \frac{1}{c}$$

$$\Rightarrow (x - a/c)^2 + y^2 = \frac{1}{c^2}$$

so fixed points are all points on semi circle of radius $\left|\frac{1}{c}\right|$ centered at $x = a/c$ on real axis (elliptic isom. composed w/ $-\bar{z}$)

② if $c = 0$

$$u(x,y) = -2ax + b = 0 \Rightarrow x = \frac{b}{2a}$$

so fixed points are all points on the vertical line $x = \frac{b}{2a}$ (parabolic isom. composed w/ $-\bar{z}$).

Classification of Orientation Preserving Isoms of $\mathbb{H}^2$:

We classify the transformations of the form $f(z) = \dfrac{az+b}{cz+d}$ $\quad a,b,c,d \in \mathbb{R}$, $ad-bc=1$ via their fixed points.

A fixed point of a transformation is a point $z_0 \in \hat{\mathbb{H}}$ s.t. $f(z_0) = z_0$.

we need to solve for z in the equation $\dfrac{az+b}{cz+d} = z$ which can be rewritten as $az+b = cz^2 + dz$ or $cz^2 + (d-a)z - b = 0$.

Thus $z = \dfrac{(a-d) \pm \sqrt{(d-a)^2+4bc}}{2c}$. Using the fact that $ad-bc=1$,

we can write $(d-a)^2+4bc = a^2+d^2-2ad+2bc+2bc$

$$= a^2+d^2-2(ad-bc)+2(ad-1) = a^2+d^2+2ad-4$$

$$= (a+d)^2-4.$$

Thus, there are 3 cases depending on whether $(a+d)^2-4$ is $\begin{matrix} >0 \\ <0 \\ =0 \end{matrix}$.

Case 1:
If $(a+d)^2-4>0$ then we have two real fixed points in $\partial\mathbb{H}^2 = \mathbb{R}\cup\{\infty\}$ call them α and β. → we call these transformations hyperbolic or loxodromic

Case 2: If $(a+d)^2-4=0$ then there is only one root and it is real, so the fixed point $\gamma \in \partial\mathbb{H}^2$ → we call these trans. parabolic

Case 3: If $(a+d)^2-4<0$ then there are two complex roots and they are complex conjugates, so there is only one fixed point whose Imaginary part is >0 and thus there is 1 fixed point in $\mathbb{H}$.
→ we call these transformations elliptic.

Claim: If z is a fixed point of f and g is a transformation that takes z to w, then w is a fixed point of $g\circ f\circ g^{-1}$.

We will use the claim to give the canonical forms of each type of trans. and show that they are isometries of $\mathbb{H}$.

Case 1: If f is hyperbolic/loxodromic and has two fixed points α and β then the trans. $g(z) = \frac{z-\alpha}{z-\beta}$ takes $\alpha \mapsto 0$, $\beta \mapsto \infty$

Note: $g(z)$ is a Möbius trans. but not nece. a hyperbolic transformation, however $g \circ f \circ g^{-1}$ is still a hyperbolic transformation.

Thus, the fixed points of $g \circ f \circ g^{-1}$ are 0 and ∞.

Since $g \circ f \circ g^{-1}$ is of the form $g \circ f \circ g^{-1}(z) = \frac{az+b}{cz+d}$ $ad-bc=1$, $a,b,c,d \in \mathbb{R}$

and 0 is a fixed point, we see that $b/d = 0 \Rightarrow b=0$.

Also ∞ is a fixed point so $\frac{a}{c} = \infty \Rightarrow c=0$.

Thus, $h(z) = g \circ f \circ g^{-1}(z) = \frac{az}{d}$ where $ad = 1 \Rightarrow d = \frac{1}{a}$.

So an associated matrix is $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ and the trans. can be written as $h(z) = a^2 z$. $a \in \mathbb{R}$. This is a dilation of $\mathbb{H}^2$!

Case 2: If f is parbolic and has a fixed point $\gamma \in \mathbb{R}$, then let $g(z) = \frac{1}{z-\gamma}$. Then $g(\gamma) = \infty$ and $g \circ f \circ g^{-1}$ has a fixed point at ∞. Thus $g \circ f \circ g^{-1}(z) = h(z) = \cancel{ad}\ \frac{az+b}{cz+d}$ has a fixed point at ∞ and so $\frac{a}{c} = \infty \Rightarrow c=0$. Thus, $h(z) = \frac{a}{d}z + \frac{b}{d}$

multiplying thru by $\frac{d}{a}$ gives $h(z) = z + b/a$ so that the canonical parabolic trans. is a translation!

Case 3: If f is elliptic and has a fixed point $\mu + i\nu \in \mathbb{H}$, then the transformation $g(z) = \frac{z-\mu}{\nu}$ takes $\mu + i\nu \mapsto i$ so that

$h(z) = g \circ f \circ g^{-1}(z) = \frac{az+b}{cz+d}$ has a fixed point of i.

Thus, $h(i) = \frac{ai+b}{ci+d} = i \Rightarrow ai+b = di - c \Rightarrow a=d$ and $b=-c$

so the condition $ad-bc=1$ becomes $a^2+b^2=1$. The matrix form of h is $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ w/ $a^2+b^2=1$. Thus, h is a rotation about i.

This is the canonical elliptic transformation of $\mathbb{H}$.

So every orientation preserving isometry of $\mathbb{H}$ is conjugate to one of these three canonical isometries.

<u>Note</u>: We still need to show that the canonical transformations are isometries.

Lengths of paths in $\mathbb{H}$ and $\mathbb{D}$:

Let $\gamma(t) = x(t) + i\,y(t)$ $t \in [a,b]$ be a path in $\mathbb{H}$.

Then the length of γ is given by

$$\int_\gamma \frac{|dz|}{\operatorname{Im}(z)} = \int_a^b \frac{|\gamma'(t)|}{\operatorname{Im}(\gamma(t))}\,dt = \int_a^b \frac{|\gamma'(t)|}{y(t)}\,dt$$

Let $\alpha(t) = x(t) + i\,y(t)$ $t \in [c,d]$ be a path in $\mathbb{D}$.

Then the length of α is given by

$$\int_\alpha \frac{2|dz|}{1-|z|^2} = \int_c^d \frac{2|\alpha'(t)|}{1-|\alpha(t)|^2}\,dt$$

EX: Find the length of the straight line between $2i$ and $1+i$ in $\mathbb{H}$.

We need a parametrization: let $\gamma(t) = (2i)(t-1) + (1+i)t$ $t \in [0,1]$.

Simplifying: $\gamma(t) = 2i \cdot t - 2i + t + i \cdot t = t + i(3t-2)$ $t \in [0,1]$

So: $\gamma'(t) = 1 + 3i \implies |\gamma'(t)| = \sqrt{1^2 + 3^2} = \sqrt{10}$

$\operatorname{Im}(\gamma(t)) = y(t) = 3t - 2$. So to find the length of γ we calculate

$$\int_0^1 \frac{\sqrt{10}}{3t-2}\,dt.$$