Math 265
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Class Handout #9

Recall that in the last class we talked about the following properties (a) (b) and 1 through 8 and concluded that if these properties hold for a set, then that set is a real vector space.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are n-vectors, then $\mathbf{u}+\mathbf{v}$ is an n-vector.

(b) If $\mathbf{u}$ is an n-vector and c is any real scalar, then $c\mathbf{u}$ is an n-vector.

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors and c and d are real scalars, then:

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
3. There exists and element $\mathbf{0}$, the zero vector, such that $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$
4. For every vector $\mathbf{u}$, there exists and element **-u** such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
5. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
6. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
7. $c(d\mathbf{u})=(cd)\mathbf{u}$
8. $1\mathbf{u}=\mathbf{u}$

Recall that we verified that all of these properties hold for $\mathbb{R}^n$, M_{mn}, P_n, P and $C(-\infty,\infty)$.

We define a new set $\mathbb{R}_n$ to be a set of all $1\times n$ matrices (look like row vectors). This set is also a vector space!

Question: Given a subset W of a vector space V, how can I tell if W is itself a vector space? If W is a vector space, we call it a *subspace* of V.

It's enough to check that properties (a) and (b) hold in W.

Exercise 1: Consider the vector space $\mathbb{R}^2$. Are the following subsets W_i subspaces of $\mathbb{R}^2$?

Let W_1 be the subset of all vectors of the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$.

Let W_2 be the subset of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ where $y \geq 0$.

Exercise 2: Consider the vector space $\mathbb{R}^3$. Is the following subset a subspace of $\mathbb{R}^3$?

Let W_4 be the subset of all vectors of the form $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$.

Exercise 3: Consider the vector space M_{33}. Are the following subsets W_i subspaces of M_{33}?

Let W_5 be the subset of all 3×3 matrices A with $\text{trace}(A) = 0$.

Let W_6 be the subset of all 3×3 matrices A with $\det(A) = 1$.

Exercise 4: Consider the vector space P_2. Is the following subset a subspace of P_2?

Let W_7 be the subset of all polynomials of the form $a_2x^2 + a_0$.

Let W_8 be the subset of all polynomials of the form $a_2x^2 + a_1x + 2$

Exercise 5: Consider the vector space $\mathbb{R}^n$. Is the following subset a subspace of $\mathbb{R}^n$?

Let W_9 be subset of all solutions to the system $A\mathbf{x} = \mathbf{0}$ where A is an $m \times n$ matrix.

The set W_9 is often called the **null space** of the matrix A, that is to say that the null space of a matrix A is the solution set to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ be vectors in a vector space V (think of V like $\mathbb{R}^n$). A vector $\mathbf{v}$ is called a *linear combination* of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ if $\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_k\mathbf{v_k}$ for some scalars $a_1, a_2, \ldots, a_k \in \mathbb{R}$.

Exercise 1: In $\mathbb{R}^3$, let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Is $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$? How about $\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$? How about $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$?

Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a set of vectors in a vector space V. The set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ is denoted by span S or $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ and is a subspace of V.

Exercise 2: Let $V = \mathbb{R}^3$. How many vectors are in span $\{\mathbf{0}\}$?

How many vectors are in span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$?

Let $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^3$. How many vectors are in span $\{\mathbf{v_1}, \mathbf{v_2}\}$?