Math 265
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Class Handout #18

Last time: Finished talking about linear transformations. Defined eigenvalues and eigenvectors and the procedure for finding them–

Procedure for finding eigenvalues and corresponding eigenvectors:
Step 1: Determine the roots of the characteristic $p(\lambda) = \det(A - \lambda I_n)$. These are the eigenvalues of A.

Step 2: For each root λ_0, find all nontrivial solutions to the homogeneous system

$$(A - \lambda_0 I_n)\mathbf{x} = \mathbf{0}.$$

That is find the null space of $A - \lambda_0 I_n$, which we call the eigenspace associated to λ_0.

For an upper triangular, lower triangular or diagonal matrix, the eigenvalues are the entries along the main diagonal.

Then we talked about a matrix $A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ whose characteristic polynomial was $p_A(\lambda) = (\lambda - 3)(\lambda^2 + 1)$. Therefore, A had complex eigenvalues $i, -i$. We will get back to our discussion of complex numbers in one class. (Note: please read Appendix B1 **before** next class)

Multiplicity of eigenvalues:

Consider $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. Then $p_A(t) = -(t+1)^2(t-3)$ and $p_B(t) = -(t+1)(t-3)^2$.

Definition: If λ is an eigenvalue for an n x n matrix A then the largest possible integer k such that $(t - \lambda)^k$ is a factor of the characteristic polynomial $p_A(t)$ is called the *multiplicity* of λ.

Theorem 1: Let λ_0 be an eigenvalue of A. The dimension of the eigenspace of A corresponding to λ_0, i.e. the dimension of Null $(A - \lambda_0 I_n)$, is less than or equal to the multiplicity k of λ_0.

Definition: Two matrices A and B are called *similar* if there exists and invertible matrix P such that $B = P^{-1}AP$.

Fact: Similar matrices have the same determinant and the same eigenvalues of the same multiplicity.

Definition: An n x n matrix A is called *diagonalizable* if $A = PDP^{-1}$ for some diagonal matrix D and some invertible matrix P. (A is similar to a diagonal matrix)

Theorem 2: An n x n matrix A is diagonalizable if and only if there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Note: This means that if we want A to be diagonalizable, A should have only real eigenvalues and no complex eigenvalues.

Theorem 3: Eigenvectors for an n x n matrix A coming from different eigenvalues are linearly independent.

Theorem 7.5: If an n x n matrix A has n DISTINCT eigenvalues, then A is diagonalizable.

For an n x n matrix A to be diagonalizable we need the following to be true:

1. A has n real eigenvalues counting multiplicity (meaning that A has no complex eigenvalues).
2. The dimension of the eigenspace for each eigenvalue λ_0, which is equal to the dimension of Null$(A - \lambda_0 I_n)$, must equal the multiplicity k of the eigenvector λ_0.

   (Note: If we are only interested in whether or not a matrix is diagonalizable then we don't need to actually find the eigenvectors for each eigenvalue. We only need to compute the nullity of $A - \lambda_0 I_n$ which just counts the number of non-pivot columns in the RREF of $A - \lambda_0 I_n$.)

Theorem 4: When A is diagonalizable, $A = PDP^{-1}$ where P is the matrix whose columns are the eigenvectors of A forming a basis for $\mathbb{R}^n$ and the diagonal entries of D are the corresponding to the columns of P (*in the correct order*).

Exercise 1: Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 3$. Since A has two distinct eigenvalues, A is diagonalizable. Find a matrix P and a diagonal matrix D such that $A = PDP^{-1}$. You should multiply out PDP^{-1} to verify that it equals A.

Exercise 2: Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. What are the eigenvalues of A? Is A diagonalizable? If so, find P and D such that $A = PDP^{-1}$.