Math 265
Professor Priyam Patel
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Class Handout #15

Exercise 1:

Let $W = \text{Span}\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ where $\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$. Apply the Gram-Schmidt process to obtain an orthogonal basis for W and then find an orthonormal basis for W.

Exercise 2: Let $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 3 \end{bmatrix}$. Write $\mathbf{u}$ as a linear combination of the orthogonal basis obtained in Exercise 1.

Exercise 3: (Discuss how you would do the following exercise, you don't need to complete it.)

Find an orthonormal basis for the subspace of $\mathbb{R}^4$ consisting of all vectors of the form $\begin{bmatrix} a - b - c \\ a \\ a - b \\ b - c \end{bmatrix}$.

Definition: A vector $\mathbf{u}$ is orthogonal to a subspace W of a vector space V if it is orthogonal to every single vector in W. The *orthogonal complement*, $W^{\perp}$, is the set of all vectors in V that are orthogonal to every vector in W.

That is, $W^{\perp} = \{\mathbf{v} \in V : \mathbf{v} \cdot \mathbf{u} = 0 \text{ for every } \mathbf{u} \in W\}$.

Note: $\mathbf{0} \in W^{\perp}$ always.

Note: $W^{\perp}$ is actually a subspace of V.

Note: $W \cap W^{\perp} = \mathbf{0}$.

Example 1:

What this suggests is that:

Theorem 5.10: Let W be a subspace of V. Then for any vector $\mathbf{v} \in V$, $\mathbf{v} = \mathbf{w} + \mathbf{u}$ where $\mathbf{w} \in W$ and $\mathbf{u} \in W^{\perp}$. We often write this as $W \oplus W^{\perp} = V$. Note that this also means that if V is n–dimensional, then dim W + dim $W^{\perp}$ = . (*Note: we will see how to compute* $\mathbf{w}$ *and* $\mathbf{u}$ *shortly.*)

Theorem 5.11: $(W^{\perp})^{\perp} = W$.

Let's try to figure out what $W^{\perp}$ is when W is one of our favorite subspaces, like the row space or column space of A.

Example 2:

Theorem 5.12: If A is an $m \times n$ matrix, then:

- is the orthogonal complement of the row space of A.
- is the orthogonal complement of the column space of A.

Procedure for finding a basis for $W^\perp$:

- Find a spanning set (or basis) for the subspace W using methods that you know. If you are given a spanning set then you can just use that or produce a basis from that spanning set.
- Put the basis vectors into the rows of a matrix A.
- Find a basis for Null A using the vector form of the solution set to $A\mathbf{x} = \mathbf{0}$.

Orthogonal Projections:

We talked last time about projecting vectors onto other vectors. Now we want to discuss projecting a vector onto a subspace W. Recall that for any $\mathbf{v} \in V$, $\mathbf{v} = \mathbf{w} + \mathbf{u}$, where $\mathbf{w} \in W$ and $\mathbf{u} \in W^\perp$. Given an orthogonal basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ for W (which we can find using the Gram-Schmidt process), then the **orthogonal projection** of $\mathbf{v}$ onto W is

$$\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}}\mathbf{w}_1 + \frac{\mathbf{v} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}}\mathbf{w}_2 + \cdots + \frac{\mathbf{v} \cdot \mathbf{w_m}}{\mathbf{w_m} \cdot \mathbf{w_m}}\mathbf{w}_m.$$

If we were given an orthonormal basis, then $\|w_i\|^2 = 1$ for all i and

$$\mathbf{w} = (\mathbf{v} \cdot \mathbf{w_1})\mathbf{w_1} + (\mathbf{v} \cdot \mathbf{w_2})\mathbf{w_2} + \cdots + (\mathbf{v} \cdot \mathbf{w_m})\mathbf{w_m}.$$

We often use the notation $\text{proj}_W(\mathbf{v}) = \mathbf{w}$. This is the closest vector in W to $\mathbf{v}$! Now, how do we find $\mathbf{u} \in W^\perp$? Recall how we did this for 2 vectors:

So $\mathbf{u} = \mathbf{v} - \mathbf{w} \in W^\perp$. Then $\mathbf{v} = \mathbf{w} + \mathbf{u}$ where $\mathbf{w} \in W$ and $\mathbf{u} \in W^\perp$

Lastly, to find the distance from $\mathbf{v}$ to W, we calculate $\|\mathbf{v} - \text{proj}_W\mathbf{v}\| = \|\mathbf{v} - \mathbf{w}\| = \|\mathbf{u}\|$.

Exercise 4: Let W be the two-dimensional subspace of $\mathbb{R}^3$ with orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$, where

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Find the orthogonal projection of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ onto W, then calculate the vector $\mathbf{u}$ and the distance from $\mathbf{v}$ to W.