Math 265
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Class Handout #13

Definition: Let A be an m x n matrix. The rows of A, considered as vectors in $\mathbb{R}_n$, span a subspace called the *row space* of A, denoted by row A. The columns of A, considered as vectors in $\mathbb{R}^m$, span a subspace called the *column space* of A, denoted by col A.

Note: We now have three subspaces associated to every matrix, its null space, its row space, and its column space.

Exercise 1: Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 7 & 1 & 8 \end{bmatrix}$. Find a basis for the column space of A.

Remember that row equivalent matrices do NOT have the same column space. However:

Theorem 4.17: If A and B are two row equivalent m x n matrices then row A = row B as subspaces of $\mathbb{R}_n$.

Exercise 2: Let $A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$. Its reduced row echelon form is $R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

1. Use the rows of R to form a basis for V = row A = row R.

2. The vector $\mathbf{v} = \begin{bmatrix} 5 & 4 & 14 & 6 & 3 \end{bmatrix}$ in in this vector subspace V. Write $\mathbf{v}$ as a linear combination of the vectors in your basis for V from part 1.

So what we have seen so far is that the non-zero rows of the reduced row echelon form of A form a basis for the row space of A. The columns of A corresponding to pivot columns in the RREF of A form a basis for the column space of A.

Definition: The dimension of row A is called the *row rank* of A. The dimension of col A is called the *column rank* of A.

Note:
row rank A = dim(Row A) = # of non-zero rows in R.

column rank A = dim(Col A) = # of pivot columns in the RREF R of A.

Claim: row rank A = column rank A.

We simply call this number the rank of A, where rank A = # of non-zero rows in R = # of pivot columns in R.

Claim: If A is an m x n matrix rank A + nullity A = ___

Exercise 3: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 4 \\ 2 & -2 & 8 \end{bmatrix}$. Thus, Row $A = \text{Span}\left\{\begin{bmatrix}1 & -2 & 3\end{bmatrix}, \begin{bmatrix}1 & -1 & 4\end{bmatrix}, \begin{bmatrix}2 & -2 & 8\end{bmatrix}\right\}$.

Find a basis for the row space of the matrix A consisting of the original row vectors of A. Compute the row rank of A.

The other way to get a basis for Row A, so that the basis is a subset of the original row vectors of A is: put rows of A in columns of a matrix– this matrix is A^T, row reduce A^T, find the pivot columns of A^T and use the corresponding row vectors of A to form a basis for Row A.

The method that you choose to use to find a basis for Row A depends on whether or not the question asks you to give a basis that is a subset of the original rows of A.

Questions:

- Is rank A = rank A^T?
- Is nullity A = nullity A^T?
- If A is a 5 x 7 matrix, what are the possible values for rank A?
- If A is a 5 x 7 matrix and rank $A = 3$, what is the dimension of the solution space to the equation $A\mathbf{x} = \mathbf{0}$?
- If A is an n x n matrix and rank $A = n$, what is the RREF of A? What other information does this give us about A?

What we have shown is that the following statements are *equivalent* for an nxn matrix A:

1. The rank of A is equal to n.
2. A is row equivalent to I_n. (The RREF of A is I_n.)
3. A is invertible (nonsingular).
4. The nullity of A is equal to zero.
5. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
6. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every n-vector $\mathbf{b}$.
7. A is a product of elementary matrices.
8. $\det(A) \neq 0$.
9. The columns of A form a linearly independent set of vectors in $\mathbb{R}^n$, and thus, span all of $\mathbb{R}^n$.
10. The rows of A form a linearly independent set of vectors in $\mathbb{R}_n$, and thus, span all of $\mathbb{R}_n$.