Math 265
Professor Priyam Patel
2/23/16

Class Handout #10

Note about subspaces: The set consisting only of the zero vector in a vector space V is a subspace of V. So for example $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$ and $\{O_{mn}\}$ is a subspace of M_{mn}.

Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ be vectors in a vector space V (think of V like $\mathbb{R}^n$). A vector $\mathbf{v}$ is called a *linear combination* of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ if $\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_k\mathbf{v_k}$ for some scalars $a_1, a_2, \ldots, a_k \in \mathbb{R}$.

Exercise 1: In $\mathbb{R}^3$, let $\mathbf{v_1} = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix}1\\0\\2\end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

Is $\begin{bmatrix}2\\4\\2\end{bmatrix}$ a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$? How about $\begin{bmatrix}-1\\-2\\2\end{bmatrix}$? How about $\begin{bmatrix}2\\1\\5\end{bmatrix}$?

Definition: Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a set of vectors in a vector space V. The span of S is the set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ and is denoted by span S or span$\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$. Additionally, span S is always a *subspace* of V.

Exercise 2: Let $V = \mathbb{R}^3$. How many vectors are in span $\{\mathbf{0}\}$?

How many vectors are in span $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$?

Let $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^3$. How many vectors are in span $\{\mathbf{v_1}, \mathbf{v_2}\}$? What can this look like geometrically?

Exercise 3: Let $V = P_2$ and let $S = \{t^2, t, 1\}$. What is span S?

Definition: If S is a set of vectors in V and span $S = V$ then said is said to *span* V or we say that V is *spanned by* S.

Example 1: Consider the following set S of 2×3 matrices

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Then span S consists of all matrices of the form $\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}$, where a, b, c, d are real numbers.

Exercise 4: Suppose A is a 5 x 5 matrix with RREF $R = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The null space of A, null A, is the solution space to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Find vectors $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^5$ such that null A = span $\{\mathbf{v_1}, \mathbf{v_2}\}$.

Exercise 5: In $\mathbb{R}^3$, let $\mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. Is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$ in span $\{\mathbf{v_1}, \mathbf{v_2}\}$?

Exercise 6: In P_2, let $\mathbf{v_1} = 2t - 1$ and $\mathbf{v_2} = t^2 + 2$. Is $\mathbf{v} = 2t^2 - 6t + 7$ in span $\{\mathbf{v_1}, \mathbf{v_2}\}$?

Exercise 7: In $\mathbb{R}^3$, let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Determine whether $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ spans $\mathbb{R}^3$. This is the same as checking whether every vector $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ is a linear combination of $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$.

Exercise 8: In P_2, let $\mathbf{v_1} = t^2 + 2t + 1$ and $\mathbf{v_2} = t^2 + 2$. Does $\{\mathbf{v_1}, \mathbf{v_2}\}$ span P_2?