MATH 351 EXAM 2 REVIEW

1. Rough outline of what we have covered– The rest of Section 1.4, Sections 2.1 and 2.2, most of Section 2.3 (except nonsingular matrices)

- Definition of a subspace of a vector space, the subspace properties, checking whether a given subset of a vector space is a subspace, the span of some vectors in a vector space is always a subspace.
- Definition of the null space of a matrix and finding the null space.
- Translation theorem
- Testing for linear dependence and independence
- Spans of linearly dependent sets and throwing out vectors–Proposition 1 Page 102
- Bases
- Dimension
- Dimension and basis theorem – Theorem 2 Page 116, Theorem 3 Page 118
- Column space and finding its basis
- Null space and finding its basis
- Row space and finding its basis
- Rank
- Nullity
- Rank-nullity theorem
- Rank and nullity in relation to transposes of matrices
- Rank of A and the solution set to the system Ax=b – Theorem 5, Theorem 7, Pages 144-145

This list is not meant to be exhaustive. I would suggest reading through all of the sections of Penney mentioned above, reviewing all definitions and making sure you understand why the Propositions and Theorems are true.

2. Practice Questions from Textbook

Section 1.4 TF: # 1, 2, 3, 4, 9, 10, 12 Exs: # 4, 11, 12, 26, 27, 28, 41
Section 2.1 TF: 2, 3, 4 Exs: 1, 3, 5
Section 2.2 TF: 1, 2, 4, 8, 9 Exs: 1, 5, 7, 8, 10, 11, 12, 14, 15, 16, 18, 19, 21
Section 2.3 TF: 4, 5, 6, 8 Exs: 2, 5, 12, 21, 22

3. Some additional practice problems

(1) Which of the following are subspaces of $\mathbb{R}^3$?

(a) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $y = x^2$.

(b) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x \geq 0$.

(c) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $z = 0$.

(d) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $3x - 2y - z = 0$.

(e) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x + y = 3$.

(2) Which of the following are subspaces of $\mathcal{P}_2$?

(a) The set of polynomials of the form $a_0 + a_2x^2$.

(b) The set of polynomials of the form $4 + bx$.

(3) Which of the following are subspaces of $M(2,2)$?

(a) The set of 2 x 2 matrices with zero trace.

(b) The set of 2 x 2 diagonal matrices.

(c) The set of 2 x 2 lower triangular matrices.

(d) The set of 2 x 2 matrices whose $(1,2)$-entry is -3.

(e) The set of 2 x 2 matrices whose $(1,1)$-entry is 0.

(4) Suppose $\mathcal{S}$ and $\mathcal{T}$ are subspaces of a vector space $\mathcal{V}$. Is $\mathcal{S} \cup \mathcal{T}$ also a subspace of $\mathcal{V}$? *(Note: this is similar to question 41 in Section 1.4 of the textbook, but we are now asking about the union of the two subspaces instead of the intersection. Think about the case where $\mathcal{V} = \mathbb{R}^n$.)*

(5) Suppose that $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. Is the span of the set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ all of $\mathbb{R}^3$? Justify your answer. *(Hint: use the dimension of* $\mathbb{R}^3$*!!)*

(6) Let $S = \left\{ \begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \right\}$? Are these four matrices linearly dependent or linearly independent?

(7) Is the set $\{t^3 + t^2 + 2t + 1, t^3 - 3t + 1, t^2 + t + 2, t + 1, t^3 + 1\}$ in $\mathcal{P}_3$ linearly dependent or independent?

(8) Are the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 5 \\ -3 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ linearly dependent or independent?

(9) For what values of c are the vectors $\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix}$ linearly independent?

(10) Find a basis for span$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ where

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -5 \end{bmatrix}, \mathbf{w}_4 = \begin{bmatrix} 2 \\ -2 \\ 3 \\ 0 \end{bmatrix}.$$

(11) Find a basis for all 2 x 2 symmetric matrices, that is the set of all $A \in M(2,2)$ such that $A = A^T$.

(12) Find a basis for the subspace $\mathcal{W} = \text{span}\{t^3+t^2+2t+1, t^3-3t+1, t^2+t+2, t+1, t^3+1\}$ of $\mathcal{P}_3$.

(13) Find a basis for the subspace $\mathcal{W} = \left\{ \begin{bmatrix} a+b+3c & 2a-b \\ 0 & 2a+b+4c \end{bmatrix} \in M(2,2) \right\}$ of $M(2,2)$.

(14) Find a basis for the subspace $\mathcal{W} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 3x + 5y + 7z = 0 \right\}$ of $\mathbb{R}^3$. What is dim $\mathcal{W}$?

(15) Find a basis for $\mathcal{W} = \text{span}\left\{\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}11\\10\\7\end{bmatrix}, \begin{bmatrix}7\\6\\4\end{bmatrix}\right\}$.

(16) Find a basis for $\mathcal{P}_6$. What is the dimension of $\mathcal{P}_6$? Explain your answer.

(17) Find a basis and the dimension for the subspace of $M(3,3)$ consisting of all 3×3 upper triangular matrices.

(18) Suppose that A is an m x n matrix. Is Null A a subspace of $\mathbb{R}^n$ or $\mathbb{R}^m$? Is Col A a subspace of $\mathbb{R}^n$ or $\mathbb{R}^m$?

(19) Find a basis for the column space and null space for the following matrix:

$$A = \begin{bmatrix}1 & -1 & 2 & 3\\2 & 6 & -8 & 1\\2 & -1 & 0 & 1\end{bmatrix}$$

What are the rank and nullity of A?

(20) Given that A has REF R, where

$$A = \begin{bmatrix}2 & -4 & 2 & -2\\2 & -4 & 3 & -4\\4 & -8 & 3 & -2\\0 & 0 & -1 & 1\end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix}1 & -2 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix},$$

find bases for the row space of A, the column space of A and the null space of A. Give the rank and nullity of A and A^T.

(21) Suppose A is an 4 x 4 matrix where rank $A = 4$. Do the columns of A form a basis for $\mathbb{R}^4$? How many non-zero rows are there in the REF of A? Does the system $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b} \in \mathbb{R}^4$? If so, is the solution unique?

(22) Suppose A is a 6 x 6 matrix where rank $A = 4$. Do the rows of A form a basis for $M(1,6)$? What is the dimension of Col A^T? How many vectors are there in any basis for Null A? How many solutions are there to the homogeneous system $A\mathbf{x} = \mathbf{0}$?

(23) Is A is a 5 x 7 matrix, what are the possible values for rank A? If A is a 7 x 3 matrix, what are the possible values of rank A? If A is a 4 x 6 matrix, what are the possible values for nullity A?

(24) If A is a 5 x 7 matrix and nulllity $A = 3$, what is nullity A^T?

(25) Classify each of the following statements as TRUE or FALSE:

(a) If $A\mathbf{x} = \mathbf{0}$ has a unique solution, then there are no vectors in Null A.

(b) If A is an m x n matrix and $n > m$, then the null space of A is not $\{\mathbf{0}\}$.

(c) If A is an m x n matrix, then dim Null A + dim Col $A = n$.

(d) If A is an m x n and rank $A < n$, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.
(e) The nullity of A always equals the nullity of A^T.
(f) The rank of A always equals the rank of A^T.
(g) If A is an m x n matrix, then dim Null A^T + dim Col $A = m$.